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DOMINANCE SOLVABLE VOTING SCHEMES

BY HERVÉ MOULIN[1]

The concept of a dominance solvable voting scheme is presented as a weakening of the strategy-proofness requirement: it relies on successive elimination of dominated strategies and generalizes the well known concept of "sophisticated voting." Dominance solvable decision schemes turn out to contain many usual voting procedures such as voting by veto, kingmaker, and voting by binary choices. The procedure of voting by elimination is proved to be an anonymous dominance solvable voting scheme which always selects an efficient alternative.

INTRODUCTION

GIBBARD'S AND SATTERTHWAITE'S ORIGINAL RESULT (see [**4, 15**]) establishing the impossibility of constructing a strategy-proof voting scheme, or game form, as soon as there are at least 3 alternatives, has generated several investigations of the following type: how does one weaken the strategy-proofness requirement so that a satisfactory voting scheme exists?[2] One possible answer is the Hurwicz-Schmeidler acceptability requirement (see [**6**]).

A voting scheme is acceptable if for every profile the Nash equilibrium set is non-empty and contained in the set of Pareto optimal outcomes. Actually acceptability does not generalize strategy-proofness. For instance, with an odd number of players and two alternatives, the simple majority vote is strategy proof but not acceptable. Another criticism is that when several Nash equilibriums lead to different Pareto optimal alternatives, the selection of one of them, which is not considered by the acceptability concept, develops among the players a battle of the sexes type conflict (called "competition for the first move" in [**12**]).

A second way to weaken the strategy-proof requirement is the "nice consistency" property of Dutta and Pattanaik [**2**] and Peleg [**14**]. This concept applies only to voting schemes in which each player's strategy is to announce an ordering of the alternatives. In this context strategy-proofness corresponds to non-manipulability, that is, to tell the truth is a Nash equilibrium. Now, a voting procedure is nicely consistent if for every profile there is at least one Nash equilibrium selecting the same alternative as the sincere n-tuple of strategies. This requirement indeed weakens the non-manipulability requirement. Its main limitation is that it is not easily extended to the general voting scheme context, however, it leads to very interesting social choice functions such as the one described by Peleg in [**14**].

In this paper we study a specific class of voting schemes, the dominance solvable voting schemes, containing the (almost empty) subclass of strategy-proof

[1] The author wishes to thank P. Levine, B. Peleg, and D. Schmeidler for fruitful remarks on the first draft of this paper. Two anonymous referees also deserve acknowledgement.

[2] Actually it is possible to keep the strategy proofness requirement either by restricting the set of feasible profiles (see Kalai and Muller [**7**], Maskin [**10**]) or by allowing the voting scheme to select lotteries on the set of alternatives (see Gibbard [**5**]). In this paper we are not concerned with these modifications of the basic voting scheme model.

voting schemes. Roughly speaking, a voting scheme is solvable if, for every profile, successive elimination of dominated strategies eventually selects a single alternative. We formalize this property by means of a pure game-theoretical concept, the dominance solution of a game, which is a particular—possibly empty—subset of the Nash equilibrium set, obtained by finitely many successive eliminations of dominated strategies.

Our concept of dominance solution relies on a long game-theoretical tradition: Luce and Raiffa's solution in the strict sense (see [**9**]), Selten's perfect equilibrium (see [**16**]), as well as Levine's strategic prejudgement (see [**8**]), are closely related to our solution concept. On the other hand, authors involved in voting theory, mainly Farquharson and Brahms [**3**, **1**] have long used the same solution concept under the names of "sophisticated voting" and "admissible strategies." Thus our approach is by no means conceptually new. What is new is the level of generality (because we use the game form formalism), from which it is easy to observe that dominance-solvable voting schemes are very numerous, including Mueller voting by veto [**13**], voting by binary choices, kingmaker procedures [**6**], etc.

Our main result is the following. For every number of alternatives and of players there exists an anonymous, dominance-solvable voting scheme which selects a Pareto optimal alternative for every profile. The simple procedure of voting by elimination (see [**3**]) provides an example: the alternatives are ranked a priori $a_1, \ldots, a_n$; a majority vote is taken to decide whether to eliminate a_1 or a_2; next a majority vote is taken to eliminate a_3 or the previous winner; and so on . . . ; after $(n-1)$ rounds of voting the selected alternative remains.

1. THE CONCEPT OF THE DOMINANCE SOLVABLE GAME

Let us consider an n-person normal form game with strategy sets $X_1, \ldots, X_n$ and pay-off functions $f_1, \ldots, f_n$ defined on $X = X_1 \times \ldots \times X_n$.[3] We denote by $f = (f_1, \ldots, f_n)$ the multi-dimensional pay-off function. Then our game is simply denoted (X, f). If for every i, Y_i is a subset of X_i, then we set $Y = Y_1 \times \ldots \times Y_n$ and we denote by $Y_{\hat{\imath}}$ the Cartesian product $Y_{\hat{\imath}} = \Pi_{j \neq i} Y_j$, with current element $y_{\hat{\imath}}$.

We will say that player i's strategy $y_i (y_i \in Y_i)$ is dominated in the subgame (Y, f) if there exists a strategy $z_i (z_i \in X_i)$ such that

$$f_i(y_i, y_{\hat{\imath}}) \leq f_i(z_i, y_{\hat{\imath}}) \quad \text{for all } y_{\hat{\imath}} \in Y_{\hat{\imath}},$$

$$f_i(y_i, y_{\hat{\imath}}) < f_i(z_i, y_{\hat{\imath}}) \quad \text{for some } y_{\hat{\imath}} \in Y_{\hat{\imath}}.$$

We denote by $D_i(Y)$ $(D_i(Y) \subset Y_i)$ the set of player i's undominated (admissible) strategies of the subgame (Y, f) and by D the mapping

$$X \supset Y = Y_1 \times \ldots \times Y_n \to D(Y) = D_1(Y) \times \ldots \times D_n(Y) \subset X.$$

Note that by definition $D(Y)$ is necessarily contained in Y. Therefore the sequence $X^0 = X, X^1 = D(X^0), \ldots, X^{t+1} = D(X^t), \ldots$ is decreasing.

[3] Actually the preference orderings defined by $f_1, \ldots, f_n$ on X only will matter. The concept and results of the whole paper are purely ordinal.

DEFINITION 1: We say that the game (X, f) is *dominance solvable* if for some $t \in \mathbb{N}$ and for every $i = 1, \ldots, n$ the pay-off function f_i is constant with respect to x_i on X^t:

$$\forall i = 1, \ldots, n \quad \forall x_i, z_i \in X_i^t \quad \forall y_{\hat{\imath}} \in X_{\hat{\imath}}^t \quad f_i(x_i, y_{\hat{\imath}}) = f_i(z_i, y_{\hat{\imath}}).$$

In this case we say that X^t is the dominance solution of the game (simply denoted d-solution; clearly if f_i does not depend on x_i over X^t and X^s with $s > t$, then $X^t = X^1$ so that the d-solution is unambiguously defined and will be denoted E instead of X^t. Thus a solution is simply obtained by successive elimination of dominated strategies. We say that the game is dominance solvable if after a finite number of steps, every player's strategies are equivalent.

PROPOSITION 1: *Suppose that for every $i = 1, \ldots, n$ the strategy space X_i is finite. Then if game (X, f) is d-solvable, its d-solution E is a subset of its set of Nash equilibriums.*

PROOF: Let $Y = Y_1 \times \ldots \times Y_n$ be any subset of $X = X_1 \times \ldots \times X_n$. We denote by $N(Y)$ the set of Nash equilibriums of game (Y, f). We will prove the following inclusion:

$$N(D(Y)) \subset N(Y).$$

Namely choose an element $x = (x_1, \ldots x_n)$ of $N(D(Y))$ and suppose that x is not a Nash equilibrium of game $(Y, f)(x \notin N(Y))$. Then there exist an integer i and a strategy $y_i \in Y_i$ such that

$$f_i(x_i, x_{\hat{\imath}}) < f_i(y_i, x_{\hat{\imath}}).$$

Now either y_i itself is undominated in game $(Y, f)(y_i \in D_i(Y))$ or there exists an undominated strategy $z_i (z_i \in D_i(Y))$ such that

$$f_i(y_i, x_{\hat{\imath}}) \leq f_i(z_i, x_{\hat{\imath}}) .$$

To prove this claim it suffices to choose any solution z_i of the maximization problem:

$$\max_{z_i \in Y_i} \sum_{z_{\hat{\imath}} \in Y_{\hat{\imath}}} f_i(z_i, z_i)$$

$$\text{subject to } f_i(y_i, x_{\hat{\imath}}) \leq f_i(z_i, x_{\hat{\imath}}),$$

and to remark that such a z_i necessarily is undominated in game (Y, f). Thus we obtain $f_i(x_i, x_{\hat{\imath}}) < f_i(z_i, x_{\hat{\imath}})$, $z_i \in D_i(Y)$ which contradicts the fact that x is a Nash equilibrium of game $(D(Y), f)$.

Applying the inclusion $N(D(Y)) \subset N(Y)$ to the sequence $X^0 = X$, $X^1 = D(X^0), \ldots$, we obtain

$$N(X) \supset N(X^1) \supset \ldots \supset N(X^t) \supset N(X^{t+1}) \ldots.$$

Suppose now that (X, f) is d-solvable and $E = X^t$ is the d-solution. In game (E, f) all strategies of any player are equivalent, therefore $N(E) = E$. This gives $E = N(X^t) \subset N(X)$ and concludes the proof. *Q.E.D.*

Examples of d-solvable games are the games where every player has (at least) a dominating strategy, and overall *the normal form of every finite extensive form game with perfect information* verifying a suitable "one-to-one" assumption. Let us make this point clear. We denote by B the finite set of "nodes" of the tree partitioned as $B = T \cup A_1 \cup \ldots \cup A_n$ where T denotes the set of terminal nodes and A_i the set of nodes where player i moves. For every a, $a \notin T$, we denote by $s(a) \subset B$, $\{s(a)\} \geqslant 2$, the nodes following node a, and make the usual assumption on s in order to obtain the tree configuration. There is a node a_0 (called the initial node) such that $s^{-1}(a_0)$ is empty; for every node a, $a \neq a_0$, $s^{-1}(a)$ contains a single node; finally a and $s^{-1}(a)$ never belong to the same set A_i.

On T are defined the pay-off functions $F_1, \ldots, F_n$. We make the following "one-to-one" assumption:

(1) $$\forall a, b \in T \quad \forall i, j = 1, \ldots, n \quad (F_i(a) = F_i(b)) \Leftrightarrow (F_j(a) = F_j(b)).$$

PROPOSITION 2: *Under assumption* (1) *the normal form game associated with the above extensive form game is d-solvable.*

PROOF: In this normal form, the strategy set of players is

$$X_i = \{\xi_i : A_i \rightarrow B / \forall a \in A_i \quad \xi_i(a) \in s(a)\}.$$

We set

$$T^{-1} = \{a \in A_1 \cup \ldots \cup A_n / s(a) \subset T\}.$$

The assumptions on s imply that T^{-1} is non-empty. Every element a of T^{-1} belongs to exactly one of the sets A_i so that we can define

$$\sigma(a) = \left\{ b \in B / F_i(b) = \sup_{a' \in s(a)} F_i\,(a') \right\} \quad \text{if } a \in T^{-1} \cap A_i.$$

We can now restrict the strategy set of every player by dropping out some dominated strategies:

$$\tilde{X}_i^1 = \{\xi_i \in X_i^0 / \forall a \in T^{-1} \cap A_i \quad \xi_i(a) \in \sigma(a)\}$$

(of course if $T^{-1} \cap A_i$ is empty, then $\tilde{X}_i^1 = X_i$).

We let the reader check that any strategy outside $\tilde{X}_i^1$ is dominated so that

$$\tilde{X}^1 = \tilde{X}_1^1 \times \ldots \times \tilde{X}_n^1 \supset D(X^0).$$

Now in view of the one-to-one assumption we have

(2) $$\forall i = 1, \ldots, n \quad \forall a \in T^{-1} \quad F_i \text{ is constant over } \sigma(a).$$

Therefore we can simplify the original tree by merging the nodes of $\sigma(a)$ for every $a \in T^{-1}$ and merging a and $\sigma^{-1}(a)$ for every $a \in T^{-1}$ (the new tree still verifies $\#\{s(a)\} \geqslant 2$ for every non-terminal node). This in turn modifies the normal form game by simply identifying equivalent strategies (see [**2**]). Clearly the new pay-off functions still verify relation (1) so that we may iterate the procedure.

Thus we define a sequence $\tilde{X}^2 \supset D(\tilde{X}^1), \ldots, \tilde{X}^{t+1} \supset D(\tilde{X}^t), \ldots$. At each step of the procedure, at least one node of the tree vanishes so that after a finite number of steps the subset $\tilde{X}_i^t$ of X_i is made up of equivalent strategies, that is to say, for every i, f_i is constant with respect to x_i on $\tilde{X}_i^t$.

In order to prove that the original normal form game is solvable, we must compare the subsets $\tilde{X}^t$ with the subsets X^t introduced in Definition 1. First of all we have $X^1 = D(X) \subset \tilde{X}^1$. Next, we remark that an n-tuple x belonging to $D(X^1)$ but not to $D(\tilde{X}^1)$ must be equivalent to a strategy of $D(\tilde{X}^1)$. (Namely, if for instance x_i is dominated in game $(\tilde{X}^1, f)$ by some $y_i \in \tilde{X}_i^1$, then we must have

$$f_i(x_i, x_{\hat{\imath}}) = f_i(y_i, x_{\hat{\imath}}) \quad \text{for every } x_{\hat{\imath}} \in X_{\hat{\imath}}^1;$$

otherwise x_i would not belong to $D_i(X^1)$. Moreover we can choose y_i belonging to $D_i(\tilde{X}^1)$.)

Thus up to identifying equivalent strategies, we have $D(X^1) = X^2 \subset D(\tilde{X}^1) \subset \tilde{X}^2$. Iterating the procedure yields $X^t \subset \tilde{X}^t$ for every t. If for some t all strategies of every player are equivalent in $\tilde{X}^t$, then the same is true for X^t. This concludes the proof of Proposition 2.

The counter-example in Figure 1 shows that the one-to-one assumption is needed.

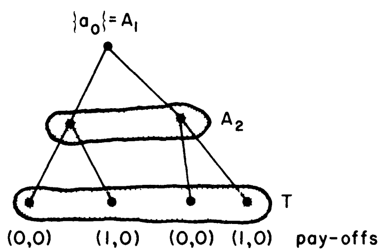

FIGURE 1

2. DOMINANCE SOLVABLE VOTING SCHEMES

DEFINITION 2: An *n-player voting scheme* is an $(n+2)$-tuple $(X_1, \ldots, X_n, A, \pi)$ where the *finite set* A is viewed as the set of alternatives, where X_i, $i = 1, \ldots, n$, is the strategy set of player i and where π is a mapping from $X_1 \times \ldots \times X_n$ into A selecting a particular alternative for every decision of the players.

In order to formalize the behavior of the players facing a particular voting scheme, we need an additional structure, namely the preference profile describing how the players order the possible decisions in A. Instead of assuming that the players' tastes are represented by an ordering of A, we will assume that each player has a strict ordering on A. This (usual) restriction avoids considerable technical difficulties without reducing severely the meaning of the results.

Let us denote by U the set of one-to-one mapping from A into $\mathbb{R}$. Every strict ordering on A thus corresponds to a subset of U, the space of utility functions. We deal with cardinal utility functions only for the matter of simplicity (as in Section 1). All concepts and results are purely ordinal.

DEFINITION 3: An n-player voting scheme $(X_1, \ldots, X_n; A, \pi)$ is said to be *d-solvable* if for every preference profile $(u_1, \ldots, u_n) \in U^n$ the game $g = (X_1, \ldots, X_n, u_1 \circ \pi, \ldots, u_n \circ \pi)$ is d-solvable. In this case the d-solution S of game g determines an unambiguous outcome, that is to say $\pi(S)$ reduces to a single outcome.

The last statement follows from Definition 2; to say that S is the d-solution of g amounts to saying that $u_i \circ \pi$ is constant with respect to x_i on S for every i. Since u_i is one-to-one, π is then constant with respect to x_i for every i.

Since a game where each player has a dominating strategy is d-solvable, then a strategy-proof voting scheme certainly is a d-solvable voting scheme. Conversely, a d-solvable voting scheme is much less attractive than a strategy-proof one. If the d-solution S is decomposed as $S = S_1 \times \ldots \times S_n$, then in order to select a strategy in S_i, player i must know the utility functions of the others players (whereas if the voting scheme is strategy-proof, the set S_i of player i's dominating strategies does not depend on u_j for $j \neq i$). On the other hand, in a world where everybody's tastes are known, a d-solvable voting scheme is a highly satisfactory way of selecting an outcome of the non-cooperative behavior of the players. When, moreover, a d-solvable voting scheme selects a Pareto optimal outcome for every utility profile, we feel that it provides a reasonable voting procedure that the examples below will justify.

A wide variety of d-solvable voting schemes is obtained by a construction that we explain now. Suppose the set A of alternatives is given and consider a tree, the nodes of which form a finite set B partitioned as $B = T \cup C$ where T denotes the set of terminal nodes; for every $a \in C$ we again denote by $s(a) \subset B$, $\#\{s(a)\} \geq 2$, the set of nodes following node a and make the assumption on s described above. To every terminal node a, $a \in T$, we associate an alternative $\tau(a) \in A$ and we assume that the mapping τ is onto.

Now for every non-terminal node a, $a \in C$, we suppose we are given a voting scheme $d^a = (X_1^a, \ldots, X_n^a; s(a), \pi^a)$ designed to select which successor of a will be chosen by the n-players (the set of players is fixed once for ever and does not vary at each node). Looking as a whole at the tree where a voting scheme is attached to each node with range in the successors of this node, we obtain a voting scheme on T, which by τ becomes a voting scheme on A. It is the voting scheme $d = (X_1, \ldots, X_n; A, \pi)$ where:

(i) $\forall i = 1, \ldots, n \quad X_i = \prod_{a \in C} X_i^a$ with current element $\xi_i : a \in C \to \xi_i(a) \in X_i^a$.

(ii) If $(\xi_1, \ldots, \xi_n) \in X_1 \times \ldots \times X_n$ is given, and if a_0 is the original node of the tree, then we consider the sequence: $a_1 = \pi^{a_0}(\xi_1(a_0), \ldots, \xi_n(a_0))$, $a_2 = \pi^{a_1}(\xi_1(a_1), \ldots, \xi_n(a_1)), \ldots$. The sequence $(a_1, a_2, \ldots)$ reaches T after finitely many steps, say at $a_k \in T$. Then we set $\pi(\xi_1, \ldots, \xi_n) = \tau(a_k)$.

We claim that *if the voting scheme d^a is d-solvable for every $a \in C$, then the voting scheme d is also d-solvable.*

The proof of this claim is exactly similar to the proof of Proposition 2, Section 1, and will be omitted. It suffices to check that for every given preference profile, the corresponding extensive form game satisfies the one-to-one assumption (1).

A dictatorial voting scheme (for some player i, the dictator, $X_i = A$ and $\pi(x_1, \ldots, x_n) = x_i$) is clearly d-solvable (even strategy-proof). By the above result, every tree configuration where a dictator is attached to each node (a different dictator for a different node) yields a d-solvable voting scheme. For instance the kingmaker (player 1, the kingmaker, chooses a dictator among the $(n-1)$ other players) or a multi-stage kingmaker (e.g., player 1 chooses a kingmaker among the $(n-1)$ other players) are d-solvable voting schemes (in [**6**] kingmaker is studied with respect to acceptability). Voting by veto can also be obtained in this manner: if A contains p outcomes and there are n players, an n-tuple of integers $(p_1, \ldots, p_n)$ is given with $p_i \geq 0$ and $\Sigma_{i=1}^n p_i = p - 1$; then voting by veto works as follows: player 1 vetoes p_1 outcomes, next player 2 vetoes p_2 among the remaining $p - p_1$ outcomes, and so on. That voting by veto is d-solvable was already noticed by Mueller in a particular case (see [**13**] for a detailed study of the d-solution).

When there are only two outcomes, strategy-proof (hence d-solvable) voting schemes do exist, for instance the direct majority vote (if n is even, one particular player, the president, is called to break ties). Therefore a tree of *binary* choices, each of which is taken by majority vote (and possibly with a different president for a different node), yields a d-solvable voting scheme. Voting by binary choices taken by majority vote is a very common assembly device and has been extensively studied in the literature (see [**3**, **1**]). In particular Farquharson's notion of sophisticated voting (see Brahms [**1**, p. 73]) corresponds exactly to our concept of d-solution.

Examples of *non-d-solvable* voting schemes are the weighted voting procedures (as soon as there are at least 3 outcomes). For instance if A contains 3 outcomes a, b, c and if there are 3 players, let us consider the plurality procedure where the weights of the players are 1.5, 1, 1. Formally this procedure is the following voting scheme:

$$X_i = A = \{a, b, c\} \quad \text{for } i = 1, 2, 3;$$

each player votes for a single outcome;

$$\pi(x_1, x_2, x_3) = x_2 \quad \text{if } x_2 = x_3 \neq x_1,$$
$$= x_1 \quad \text{otherwise.}$$

We call player 1 the president since his vote has more weight.

We prove that the voting scheme (X_1, X_2, X_3, A, π) is not d-solvable by considering the following utility profile:

player 1 (the president)	$a > b > c$,
player 2	$b > c > a$,
player 3	$c > b > a$.

Whatever the preferences of the other players are, player 1's dominating strategy is to vote a. Then player 2 and player 3 can only eliminate strategy a as being dominated by any other vote. In order to decide whether they will vote b or c, they face the following 2×2 subgame:

$$\text{player 2}\begin{cases}\text{vote } b \\ \text{vote } c\end{cases}\begin{array}{|c|c|}\hline b & a \\ \hline a & c \\ \hline \end{array}$$

$$\underbrace{\text{vote } b \quad \text{vote } c}_{\text{player 3}}$$

In this subgame, which is of the battle of sexes type, no player has a dominated strategy but rather each player tries to commit himself to vote for his own favorite, implying that the original game is not d-solvable. Note that if the Condorcet intransitivity arises, then the associated game by the above procedure is d-solvable, leading to the counter-intuitive effect that the president's least preferred outcome is elected by the others coalesced. (See Farquharson [**3**].)

3. ANONYMITY, NEUTRALITY, EFFICIENCY

In contrast with the requirement of strategy-proofness, the d-solvability requirement allows for a wide variety of voting procedures. We will now investigate the compatibility of d-solvability with 3 basic properties of voting schemes: anonymity, neutrality, and efficiency.

A d-solvable voting scheme $(X_1, \ldots, X_n; A, \pi)$ associates to every preference profile $(u_1, \ldots, u_n) \in U^n$ a solution outcome denoted $S(u_1, \ldots, u_n) \in A$ (see Definition 3). The 3 properties of d-solvable voting schemes are now defined in terms of the "social choice function" S: (a) S is *anonymous* iff it is symmetric with respect to $(u_1, \ldots, u_n)$; (b) S is *neutral* if for every bijection θ of A into itself we have:

$$S(u_1 \circ \theta, \ldots, u_n \circ \theta) = \theta(S(u_1, \ldots, u_n));$$

(c) S is *efficient* if $S(u_1, \ldots, u_n)$ is a Pareto optimal outcome of A.

The kingmaker procedure, as well as voting by veto are neutral (i.e. fair with respect to outcomes) but not anonymous. On the other hand, a procedure described as a tree of binary choices taken by majority voting (see Section 2) is anonymous (i.e., fair with respect to players) when n is odd (since no president is needed) but not neutral.

The kingmaker procedure, and voting by veto are efficient. Let us prove for instance that *voting by veto is efficient.* We suppose that the integers $p_1 \geqslant 0, \ldots, p_n \geqslant 0$ such that $\Sigma_{i=1}^n p_i = p-1$ are given as well as the utility profile $(u_1, \ldots, u_n) \in U^n$.

The computational algorithm leading to the solution outcome is the following (see [**13**]): Let $A_n \subset A$ be the set of the p_n least preferred outcomes of player n. Let $A_{n-1} \subset A \backslash A_n$ be the set of the p_{n-1}-least preferred outcomes of player $(n-1)$ among the outcomes of $A \backslash A_n \ldots$. Let $A_1 \subset A \backslash (A_n \cup A_{n-1} \cup \ldots \cup A_2)$ be the set of the p_1-least preferred outcomes of player 1 among the outcomes of $A \backslash (A_n \cup \ldots \cup A_2)$. Then the remaining outcome $(A \backslash \bigcup_{i=1}^n A_i)$ is $S(u_1, \ldots, u_n)$. Justifications of this backward induction can be found in [**13**].

Let us set $a = S(u_1, \ldots, u_n)$. Every outcome b, $b \neq a$, belongs to some A_i. By construction of A_i, b is then less preferred than a by player i. Therefore b does not dominate a and a is a Pareto optimal outcome.

Thus voting by veto and the (multi-stage) kingmaker give attractive examples of d-solvable voting schemes which are *neutral and efficient.* Of course a dictatorial voting scheme is enough to prove compatibility of neutrality and efficiency for d-solvable voting schemes. But voting by veto is more attractive in the sense that player i can be guaranteed that the selected outcome does not belong to his p_i-least preferred outcomes; hence the "security levels" of the players in voting by veto dominate those of the players in a dictatorial voting scheme.

As we have seen in Section 2, every voting scheme that is described as a tree of binary choices taken by majority voting is d-solvable, but not every such procedure is efficient. Suppose for instance $\#(A) = 4$. Consider the tree of binary choices in Figure 2 (it is an "amendment" procedure according to Farquharson's terminology).

If there are 3 players with the following preferences:

player 1 $\quad u_1(a_3) > u_1(a_4) > u_1(a_1) > u_1(a_2)$,

player 2 $\quad u_2(a_4) > u_2(a_1) > u_2(a_2) > u_2(a_3)$,

player 3 $\quad u_3(a_2) > u_3(a_3) > u_3(a_4) > u_3(a_1)$,

then the sincere majority vote leads to:

a_3 defeats a_4, a_2 defeats a_3, a_1 defeats a_2.

This implies that the d-solution is a_1, although a_4 is unamimously preferred to a_1. We prove now that anonymity and efficiency are two compatible requirements for a d-solvable decision scheme, and can be obtained by a suitable tree of binary choices taken by majority voting.

THEOREM: *For any n and any finite A, there exists a d-solvable anonymous and efficient voting scheme* $(X_1, \ldots, X_n, A, \pi)$.

The following voting scheme called "voting by eliminating" satisfies the above requirements: Rank the outcomes $a_1, \ldots, a_p$. Take a majority vote between

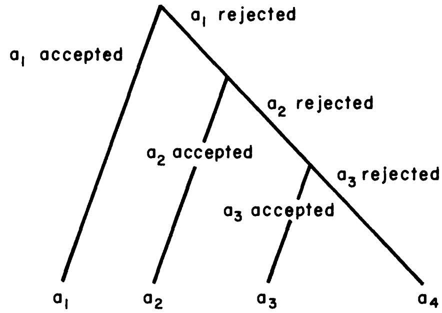

FIGURE 2

"eliminate a_1" and "eliminate a_2" (in case n is even break possible ties by eliminating a_1); denote b_2 the winner. Take a majority vote between eliminate b_2 and eliminate a_3 (break possible ties by eliminating b_2); denote b_3 the winner. And so on (at each step the ties are broken in favor of the new competitor). The selected alternative is b_p.

When A contains 3 or 4 alternatives we obtain the tree in Figure 3.

Since voting by eliminating is a tree of binary choices taken by majority votes we already know that it is a d-solvable voting scheme (see Section 2). Anonymity is clear so it remains to prove efficiency.

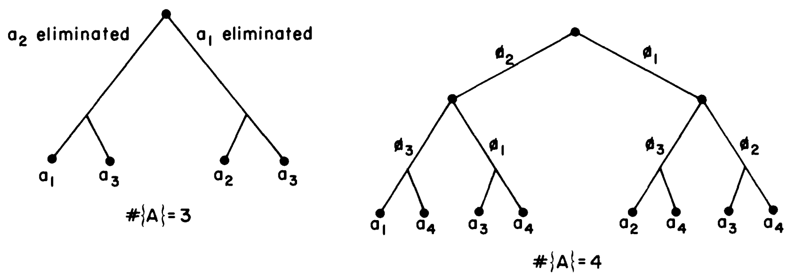

FIGURE 3. Ties are broken by going *right*.

In order to prove efficiency of voting by eliminating, we must describe the computational algorithm leading to the d-solution. Let us assume that the preference profile $(u_1, \ldots, u_n)$ is given. Then we denote by m the choice function induced by majority vote:

$$\forall a, a' \in A \quad m(a, a') = a \quad \text{if } \#\{k/u_k(a) \geq u_k(a')\} \geq \frac{n}{2},$$

$$m(a, a') = a' \quad \text{if } \#\{k/u_k(a') \geq u_k(a)\} > \frac{n}{2}.$$

Voting by eliminating is made up of $(p-1)$ rounds of voting. At the $(p-1)$th round of voting, an alternative a_i, $1 \leqslant i \leqslant p-1$, competes with a_p. If players use undominated strategies, the winner is denoted $\varphi^1(a_i)$:

(2) $$\varphi^1(a_i) = m(a_p, a_i) \quad \text{for every} \quad i \leqslant p-1.$$

At the $(p-2)$th round of voting, an alternative a_i, $1 \leqslant i \leqslant p-2$, competes with a_{p-1}. Given that the players use undominated strategies at the $(p-1)$th round, then undominated voting at the $(p-2)$th round yields the winning alternative $\varphi^2(a_i)$:

$$\varphi^2(a_i) = m(\varphi^1(a_{p-1}), \varphi^1(a_i)) \quad \text{for every} \quad i \leqslant p-2.$$

And so on .... At the $(p-k)$th round of voting, alternative a_i, $1 \leqslant i \leqslant p-k$ competes with a_{p-k+1}. The alternative $\varphi^k(a_i)$ is simply the d-solution of the voting by eliminating procedure restricted to the last k rounds of voting and starting with a_i:

(3) $$\varphi^k(a_i) = m(\varphi^{k-1}(a_{p-k+1}), \varphi^{k-1}(a_i)) \quad \text{for every} \quad i \leqslant p-k.$$

Finally, at the first round of voting, a_1 competes with a_2 and the d-solution is $\varphi^{p-1}(a_1)$ given by:

(4) $$\varphi^{p-1}(a_1) = m(\varphi^{p-2}(a_2), \varphi^{p-2}(a_1)).$$

We have to prove that $\bar{a} = \varphi^{p-1}(a_1)$ is a Pareto optimal alternative. First of all, in view of the above algorithm, $\bar{a}$ can be written as $\varphi^1(a_i)$ for some a_i, $1 \leqslant i \leqslant (p-1)$; therefore (2) implies that $\bar{a}$ either is a_p or defeats it strictly by (sincere) majority voting. Then $\bar{a}$ is not Pareto dominated by a_p.

Now fix an integer k, $1 \leqslant k \leqslant p-1$. We will prove that $\bar{a}$ is not Pareto dominated by a_{p-k}.

(i) *Suppose first* $\varphi^k(a_{p-k}) = a_{p-k}$. If $k = (p-1)$, this equality means $\bar{a} = \varphi^{p-1}(a_1) = a_1$ and our claim is proved. If $1 \leqslant k < (p-1)$, then the above algorithm shows that $\bar{a}$ can be written as $\varphi^{k+1}(a_i)$, for some i, $1 \leqslant i \leqslant p-k-1$. In view of equation (3) we obtain:

$$\bar{a} = \varphi^{k+1}(a_i) = m(\varphi^k(a_{p-k}), \varphi^k(a_i)) = m(a_{p-k}, \varphi^k(a_i)).$$

Thus $\bar{a}$ either is a_{p-k} or defeats it strictly by (sincere) majority voting. In every case $\bar{a}$ is not Pareto dominated by a_{p-k}.

(ii) *Now suppose* $\varphi^k(a_{p-k}) \neq a_{p-k}$. Note that if we set $\varphi^0(a) = a$ for every a, then formula (2) becomes a particular case of formula (3). Using this convention, there exists at least one integer l, $1 \leqslant l \leqslant k$, such that:

(5) $$\begin{cases} \varphi^{l-1}(a_{p-k}) = a_{p-k}, \\ \varphi^l(a_{p-k}) \neq a_{p-k}. \end{cases}$$

Denote $b = \varphi^{l-1}(a_{p-l+1})$. Then equation (3) combined with (5) gives:

$$\varphi^l(a_{p-k}) = m(b, \varphi^{l-1}(a_{p-k})) = m(b, a_{p-k}) = b.$$

Therefore,

$$(6) \qquad \#\{k/u_k(b) > u_k(a_{p-k})\} \geqslant \frac{n}{2}.$$

(Note that since $b \neq a_{p-k}$, $u_k(b) \geqslant u_k(a_{p-k})$ is equivalent to $u_k(b) > u_k(a_{p-k})$.)

On the other hand, the algorithm (2), (3), (4) shows that $\bar{a}$ can be written as $\varphi^l(a_i)$ for some i, $1 \leqslant i \leqslant p-1$. By (3) we have

$$\bar{a} = \varphi^l(a_i) = m(b, \varphi^{l-1}(a_i)).$$

Thus $\bar{a}$ either is b or defeats it strictly by (sincere) majority voting; if $\bar{a} = b$ then by (6) $\bar{a}$ is not Pareto dominated by a_{p-k}; if $\bar{a} \neq b$ then we have

$$(7) \qquad \#\{k/u_k(\bar{a}) > u_k(b)\} > \frac{n}{2}.$$

Inequalities (6) and (7) together yield that for at least one player k, $u_k(\bar{a}) > u_k(a_{p-k})$ which in turn proves that $\bar{a}$ is not Pareto dominated a_{p-k}. *Q.E.D.*

In order to illustrate the social choice function associated with voting by eliminating we comment on formulas (2), (3), and (4). Suppose n is odd and that alternative a is a Condorcet winner. Then a is the d-solution of the voting by eliminating procedure. Suppose a is ranked kth; $a = a_k$. By induction on (2) and (3) we obtain:

$$\varphi^i(a_k) = a_k \quad \text{for every } i \text{ such that } i \geqslant k,$$

$$\varphi^l(a_i) = a_k \quad \text{for every } i, l \text{ such that } i \leqslant l < k.$$

Suppose now that the players vote sincerely (or myopically); that is, at each round of voting each player votes according to his own preference between the two competing alternatives. Then the Condorcet winner a is also selected. In a case where there is a Condorcet winner, one cannot distinguish sincere voting from sophisticated voting. The situation is very different when there is no Condorcet winner. In this case one checks that the last ranked outcome a_p is never selected as the d-solution (we let the reader check from (2), (3), and (4) that $a_p = \varphi^{p-1}(a_1)$ if and only if a_p is a Condorcet winner—we still assume that n is odd) whereas the first ranked outcome a_1 is never selected by sincere voting (since a_1 is selected as the d-solution (we let the reader check from (2), (3), and (4) that speaking, sincere voting puts the first ranked alternatives at a disadvantage whereas sophisticated voting (that is the d-solution) puts the last ranked alternatives at a disadvantage.

In case n is even, the same comments apply with the special feature that the binary relation representing the majority vote between two alternatives takes into account their respective rank:

(8) if $i < j$, then the majority prefers a_i to a_j if and only if $a_i = m(a_j, a_i)$.

In the voting procedure considered, we have definitely favored the last ranked alternative. But we could also favor the first ranked alternative by considering the following procedure: at the kth round of voting a majority vote is taken between $a_i = b_k$ and a_{k+1}. Possible ties are broken by eliminating a_{k+1}.

This modifies the binary relation (8) into:

(9) if $i < j$ then the majority prefers a_i to a_j if and only if $a_i = m(a_j, a_i)$.

As a matter of fact, the new voting scheme that we obtain is still d-solvable, anonymous, and efficient. The proof of this claim is word for word similar to the above proof and will be omitted.

The main property of our voting by eliminating procedure is the efficiency of the d-solution: This is not true of sincere voting in the same procedure. Suppose for instance $\#A = 3$, $n = 2$, and that the preference profile corresponds to the situation depicted in Figure 4. We use the voting scheme described in the statement of the theorem. Ties are broken in favor of the last ranked alternative. If players vote sincerely, then a_2 defeats a_1 (there is a tie). Next a_3 defeats a_2. On the contrary the d-solution is a_1: the players are aware that a_3 will defeat a_2 in the

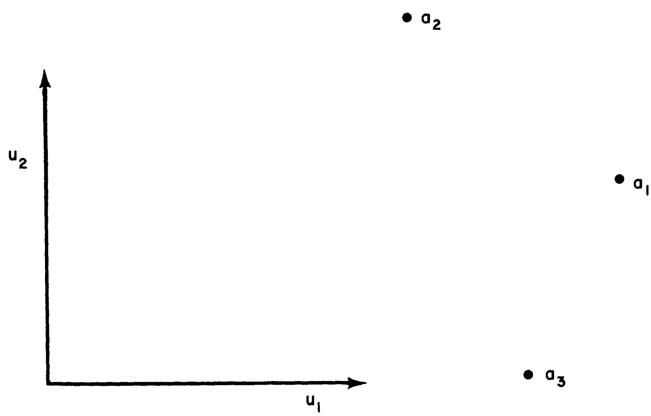

FIGURE 4

second round of voting so that voting for a_1 at the first round is a dominating strategy for each of them.

As a final remark we note that there are other d-solvable voting schemes yielding an anonymous and efficient social choice function . For instance, if there are 3 alternatives and 2 players then the following voting scheme shares all these properties:

a_1	a_1	a_1	a_1	a_1	a_1
a_1	a_3	a_1	a_3	a_1	a_3
a_1	a_1	a_2	a_2	a_2	a_2
a_1	a_3	a_2	a_2	a_2	a_2
a_1	a_1	a_2	a_2	a_3	a_3
a_1	a_3	a_2	a_2	a_3	a_3

where X_1 corresponds to the rows; X_2 corresponds to the columns; $\pi(x_1, x_2)$ is located at the intersection of the corresponding row and column.

To determine which social functions can be obtained by means of a d-solvable voting scheme is a wide open problem. This question is precisely formulated as: given a SCF that maps Φ from U^n into A, does a d-solvable voting scheme $(X_1, \ldots, X_n, \pi)$ exist such that the corresponding SCF is Φ, that is, such that the d-solution of game $(X_1, \ldots, X_n, u_1 \circ \pi, \ldots, u_n \circ \pi)$ selects the alternative $\Phi(u_1, \ldots, u_n)$ for every profile $(u_1, \ldots, u_n)$?

In particular for some value of $\#\{A\}$ and n one can easily construct anonymous, neutral and efficient SCF's.[4] An open question is whether or not such an SCF can be obtained as the d-solution of a d-solvable voting scheme. Other open questions concern the relation of d-solvability with other approaches to voting behavior. Do there exist, for instance, acceptable and d-solvable voting schemes, consistent and d-solvable voting schemes?

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[4] This construction is possible only if n has no prime factor less than or equal to $\#\{A\}$. Namely if n is no larger than $a = \#\{A\}$, then consider an $n \times n$ latin square filled with n outcomes among the a outcomes; in the corresponding profile where these n outcomes are unanimously preferred to all other alternatives, it is clear by efficiency that any social choice violates anonymity. If $n > a$, decompose the players into r blocks of equal size, if possible, with $l < r \leq a$. The same profile as before, but on the r blocks, again shows there is no social choice.

When $\#\{A\} = 3$, $n = 5$, one can construct easily an SCF with the above properties. But the author failed to decide whether or not such an SCF can be obtained by means of a d-solvable voting scheme.

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