ENDOGENOUS ASYMMETRY IN SEQUENTIAL AUCTIONS[1]

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Abstract

This paper proposes an innovative methodology for sequential auctions of homogeneous goods that creates asymmetry among participants, thereby achieving higher revenue for the auctioneer. The asymmetry arises endogenously from a competitive advantage in the second auction granted to the winner of the first auction. The analysis shows that the auctioneer's expected revenue in this scenario is higher than in auctions without a competitive advantage and approaches the expected revenue of an optimal reserve price auction and strictly exceeds that revenue if there are at least four participants. After analyzing the benefits and drawbacks of the competitive advantage mechanism, this paper concludes that it represents a more effective, revenue-enhancing auction design that circumvents the Revenue Equivalence Theorem in the traditional paradigm of private values with uniform value distributions.

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1. INTRODUCTION

In any country, the public sector is one of the largest consumers of goods and services. According to McHugh (2020), a typical American school consumes 320,000 sheets of paper per year; Nassar et al. (2022), calculates the number of prescriptions in England and Wales above 1,918,138 for every 100,000 inhabitants in 2019. Just as the purchase of paper for a public school and the purchase of medicines for a public hospital, a large part of public procurement of goods and services is implemented through competitive auction mechanisms, which have a repetitive character. Thus, the choice of the optimal mechanism for acquiring these materials must take into account their recurring nature.

Since the seminal paper by Vickrey (1961), the academic literature has successfully developed the theory of static auctions for a single object, which is now well understood, at least under the standard paradigm of independent private values (Myerson, 1981). The extension to dynamic auction models, on the other hand, is less standard.

When considering an infinitely repeated relationship, a significant number of studies continues to seek conditions to validate the "Folk Theorem," in which a cooperative behavior that yield Pareto superior outcomes is reached, as in the seminal works of Fudenberg and Maskin (1986) and Abreu (1988), among many contemporaneous and subsequent studies. Viewed solely in the context of the players themselves, the Folk Theorem presents trigger strategies that generate highly beneficial outcomes for all, which are unattainable in a static context. However, in the case of an oligopoly, what is beneficial to the players induces an outcome equivalent to a monopoly, which can be detrimental to consumers, bringing about deadweight losses to the economy. Thus, the study of repeated games highlights the inevitability of collusion, which limits competition among oligopolistic firms aware of their long-term interactions, as extensively reviewed by Sorin (1992). Consequently, the Folk Theorem in repeated games has driven research in Public Economics, focusing on developing mechanisms to dismantle cartels of firms with market power. The attractiveness of coalitions among players and the search for mechanisms to limit them are issues that have also captured the attention of auction theory and applied researchers, particularly in the literature known as "bidding rings" (Aoyagi, 2003; Marshall & Marx, 2012; McAfee & McMillan, 1992; Phillips et al., 2003; Skrzypacz & Hopenhayn, 2004).

A more recent strand of the literature considers finite sequential auctions, typically two auctions, and seeks to understand how this dynamic relationship alters the equilibria found in the static version of the games. A key focus of this literature is to determine whether the mechanism design choice of selling multiple objects in two sequential auctions, as opposed to a single simultaneous auction, leads to an increase in the auctioneer's expected return. For instance, Salant & Cabral (2019) find that in the case of unit demand by risk-neutral participants, sequential selling is beneficial for the auctioneer. Similar results are found in Charkraborty (2018) for the case of multiple-unit demand and in Charkraborty (2019) for the case of unit demand with risk-averse participants.

Despite being extremely rich, this literature does not apply to situations where there is no option for simultaneous sales, as in the case of recurring public procurements. Another line of research assumes that objects are sold sequentially and tries to understand how this auction format affects participant behavior when there is a relationship between the goods auctioned in the first and second auctions. De Silva et al. (2005) focus on the context of construction service bidding and argue that winning a bid yields a cost reduction if the same bidder wins the subsequent bid, due to economies of scale, for instance. In this case, the winner of the first auction bids more aggressively, benefiting the auctioneer. Similarly, Jofre-Benet & Pesendorfer (2014) consider the possibility that the goods sold in sequential auctions may be complements or substitutes, concluding that in the case of substitute goods, the first-price format is more desirable, and if the goods are complements, the second-price format is more desirable from the auctioneer’s expected revenue perspective.

In the examples mentioned above, there is an asymmetry between the participants of the second auction depending on who won the first. This asymmetry, however, is generated exogenously and is due to a relationship between the good sold in the first auction and the one sold in the second, which affects the expected benefit of winning the second auction for the winner of the first auction, thus benefiting the auctioneer.

However, when there is no synergy between the two goods, there is no exogenous asymmetry that can favor the auctioneer. The goal of this paper is precisely to propose a mechanism to be applied to sequential auctions without synergies that has the effect of increasing the auctioneer's expected revenue. The basic principle of the mechanism is to endogenously create asymmetry among the participants, thereby making their bids more aggressive. The mechanism, applied to two consecutive auctions, introduces asymmetry in the second auction by granting a competitive advantage to the winner of the first auction. By introducing this advantage, the auction design increases the interest of all participants in winning the first auction, as the benefits of victory are now enhanced: in addition to receiving the item, the winner has an increase in her chance of winning the second auction. With the augmented interest in winning the first auction, all participants raise their bids, thereby increasing the auctioneer's expected revenue in the first auction and generating a higher expected revenue across both auctions than would be obtained by simply repeating the static auction without introducing the asymmetry-generating rule.

The literature on auctions has extensively researched a different type of asymmetry: the exogenous differences between participants. These differences may arise from an asymmetric ex-ante distribution of values (Kaplan & Zamir, 2012; Kirkegaard, 2022), or from participants having differential information about the value of the auctioned object (Wilson, 1966; Wilson, 1967; Milgrom & Webert, 1982; Xu & Cavallo, 2022). In the present model, the participants are ex-ante identical, and the asymmetry arises endogenously from the auction design.

In addition to this introduction, this paper is organized as follows. Section 2 first defines the basic elements of the model of independent private values with two sequential auctions. It then presents the symmetric Bayesian Nash equilibria as well as the auctioneer's expected revenue for the standard cases where no reserve price is used and where the reserve price that maximizes the auctioneer's revenue is included. According to auction theory, this latter mechanism is the one that maximizes the auctioneer's revenue among all possible standard auctions (Krishna, 2010, chapter 5). Section 3 proposes the new mechanism, finds the corresponding Nash equilibrium for the sequential auctions, and calculates the auctioneer's expected revenue for two possible implementations of the competitive advantage mechanism. Section 3 also discusses the relative advantages of the proposed mechanism. Finally, Section 4 presents concluding remarks along with proposals for extensions of the proposed model.

2. THE BASIC MODELS AND THEIR CLASSIC SOLUTIONS

This section describes the two basic auction models that will be compared with our proposed mechanism There are n risk neutral agents (players, auction participants or bidders) who wish to acquire as many units as possible of a homogeneous good. In total, two units are sold in two consecutive auctions. The value that agent $i = 1, \dots, n$ assigns to the good is a random variable v_i that is distributed in the interval $[0,1]$ according to the same probability distribution $F(v_i)$ and is realized at each new auction. The agents' values are private, symmetric and independent in each auction and across different periods. The objects sold have no value to the auctioneer, who derives all their utility from the sales. For the sake of simplicity, we suppose agents do not discount the future.

2.1. Two consecutive and independents auctions

Suppose two independent auctions are conducted, one after the other, without any link between them. Then, from an ex-ante perspective, the solution will be the same for each auction. According to the Revenue Equivalence Theorem (Myerson, 1981), the auctioneer's revenue will be the same regardless of the format chosen from the four traditional formats: first price sealed-bid, second price sealed-bid (Vickrey, 1961), ascending open, and descending open auctions. Consider the commonly used first-price sealed-bid auction. In this auction, each player submits a bid in a sealed envelope to the auctioneer, who opens them and selects the highest bid as the winner. The winner pays his bid. Let us name $\mathcal{M}_1$ this basic model.

Definition 1. *Model $\mathcal{M}_1$: two sequential first price sealed-bid auctions without a reserve price.*

The following proposition presents the symmetric Bayesian Nash equilibrium strategies of the game as well as the expected revenue of the auctioneer.

PROPOSITION 1. *In model $\mathcal{M}_1$ the symmetric Bayesian Nash equilibrium strategies of the players in each auction are:*

$$\beta_{tk}(v) = v - \frac{1}{F(v)^{n-1}} \int_0^v F(x)^{n-1} dx\,, \quad t = 1,2;\ k = 1, \dots, n$$

Where $t = 1,2$ corresponds to the first ($t = 1$) or second ($t = 2$) auction and k corresponds to the player. Furthermore, the expected revenue of the auctioneer in this sequential auction is:

$$R_1(n) = 2R_{11}(n) = 2\int_0^1 n(n-1)xF(x)^{n-2}[1 - F(x)]f(x)dx$$

In particular, when the values of the bidders are uniformly distributed in $[0,1]$, i.e., $F(v) = v$, the corresponding bid function and expected revenue are, respectively:

$$\beta_{tk}(v) = \frac{n-1}{n}v, \quad t = 1,2;\ k = 1, \dots, n; \qquad R_1(n) = 2R_{11}(n) = 2\frac{n-1}{n+1}$$

Where $R_{11}(n) = \frac{n-1}{n+1}$ is the expected revenue in each one of the two identical auctions.

Proof: See Krishna (2002, chapter 2) or Menezes and Monteiro (2004, chapter 3).

2.2. Using a reserve price

Now, suppose two independent first-price auctions are conducted, one after the other, but the auctioneer sets a reserve price r, below which the item will not be sold. In this case, by choosing the reserve price optimally, the auctioneer can increase their expected revenue in each auction (Krishna, 2002; Ausubel & Cramton, 2004). Let us name $\mathcal{M}_2(r)$ this model.

Definition 2. *Model $\mathcal{M}_2(r)$: two sequential first price sealed-bid auctions with reserve price $r > 0$.*

Proposition 2 presents the symmetric Bayesian Nash equilibrium strategies of the game with reserve prices, as well as the expected revenue of the auctioneer.

PROPOSITION 2. *In model $\mathcal{M}_2(r)$ the symmetric Bayesian Nash equilibrium strategies of the players in each auction are:*

$$\beta_{tj}(v;r) = v - \frac{1}{F(v)^{n-1}}\int_r^v F(x)^{n-1}dx \quad if \quad v \geq r$$

$$\beta_{tj}(v;r) \in [0,r) \quad if \quad v < r$$

Where $t = 1{,}2$ corresponds to the first ($t = 1$) or second ($t = 2$) auction and $j = 1, \dots, n$ corresponds to the player. Furthermore, the expected revenue of the auctioneer in this sequential auction is:

$$\begin{aligned} R_2(r;n) &= 2R_{21}(r;n) \\ &= 2nr\big(1 - F(r)\big)F(r)^{n-1} + 2n(n-1)\int_r^1 x\big(1 - F(x)\big)f(x)F(x)^{n-2}dx \end{aligned}$$

In particular, when the values of the bidders are uniformly distributed in $[0,1]$, i.e., $F(v) = v$, the corresponding bid function and expected revenue are, respectively:

$$\beta_{tk}(v;r) = \frac{n-1}{n}v + \frac{r^n}{v^{n-1}} \quad if \quad v \geq r$$

$$\beta_{tk}(v;r) \in [0,r) \quad if \quad v < r$$

$$R_2(r;n) = 2R_{21}(r;n) = 2\frac{n-1}{n+1} + 2r^n - 4\frac{n}{n+1}r^{n+1}$$

Where $R_{21}(r;n) = \frac{n-1}{n+1} + r^n - 2\frac{n}{n+1}r^{n+1}$ is the expected revenue in each one of the two identical auctions.

Proof: See Krishna (2002, chapter 2) or Menezes and Monteiro (2004, chapter 3).

The following results is a direct consequence of maximizing the revenue $R_2(r)$.

Corollary 1. *The reservation price that maximizes the expected revenue of the seller* *does not depend on the number of participants and is given by the implicit solution of the following equation:*

$$r^* = \frac{1 - F(r^*)}{f(r^*)}$$

In particular, if the values of the bidders are uniformly distributed in $[0,1]$*:*

$$r^* = \frac{1}{2}$$

And the corresponding maximized expected seller's revenue is:

$$R_2\left(\frac{1}{2}; n\right) = 2\frac{n-1}{n+1} + \frac{1}{2^{n-1}}\frac{1}{n+1} = 2\left(R_{11}(n) + \frac{1}{2^n}\frac{1}{n+1}\right)$$

Proof: See Krishna (2002, chapter 2) or Menezes and Monteiro (2004, chapter 3).

When comparing the result obtained with the optimal reserve price to that from the standard uniform distribution model, it becomes apparent that the auctioneer's expected revenue has increased by 1/6, corresponding to a significant gain of 25% over the previous revenue, when there are only two players. But this benefit reduces drastically as the number of players increases, reaching a mere 2% with four players.

Furthermore, this benefit does not come without cost. Indeed, when $r = \frac{1}{2}$, the object is not sold with probability $\left(\frac{1}{2}\right)^n$. When there are only two players this corresponds to one fourth of the times. This property of Pareto inefficiency of auctions with reserve prices is particularly troubling when considering their application in situations where the public sector is determined to ensure the auction's success, i.e., ensuring that the item (such as a public company) is necessarily sold. For instance, in the case of the uniform distribution, if the cost of the auction failure is 0.2, them if there are at least 4 players the accrued revenue with the optimal reservation price does not compensate for the expected cost of not selling the good[4]. Next, we propose a mechanism that has the potential to increase the expected revenue of the auctioneer while, most importantly, ensuring that the object is sold.

3. INTRODUCING ENDOGENOUS ASYMMETRY AMONG THE PLAYERS

Consider now the two original sequential auctions without a reserve price, where the first auction is a standard first-price auction but include in the mechanism an asymmetry-

[4] The accrued expected revenue in one auction is $\frac{1}{2^n}\frac{1}{n+1}$ whereas the expected cost of failure is $\frac{1}{2^n}c$ where c is the actual cost of failure.

creation rule to be specified later, valid for the second auction.

The asymmetry-creation rule gives a competitive advantage in the second auction to the winner of the first auction. Such competitive advantage, in turn yields a higher expected payoff off in the second auction to the winner of the first auction, in comparison to a loser of the first auction.

Let $u_w(n) = u_w(n, \beta_2)$ be the ex-ante expected utility in the second auction of the winner of the first auction, i.e. before that player observes his value in the second auction, given the predicted Perfect Bayesian equilibrium β_2 of the second auction. Similarly, be $u_l(n) = u_l(n, \beta_2)$ be the ex-ante expected utility in the second auction of a loser of the first auction, given the second auction predicted Perfect Bayesian equilibrium β_2. For simplicity, we drop the reference to the equilibrium β_2.

By construction, $u_w(n) > u_l(n)$. Let $\Delta(n) = u_w(n) - u_l(n)$. Then, $\Delta(n)$ is the expected additional return in the second auction to a winner of the first auction. Call this expected additional return the “competitive advantage” of the winner. Denote by $\mathcal{M}_3$ the corresponding selling mechanism with competitive advantage.

Definition 3. *Model $\mathcal{M}_3$: two sequential sealed-bid auctions without a reserve price, the first auction is a first price sealed bid auction. The second auction includes an asymmetry-creation rule that yields a competitive advantage $\Delta(n)$ to the winner of the first auction, where n is the number of players.*

Note that the previous definition is purposefully not explicit about the specific format of the second auction. The following proposition shows that only the competitive advantage $\Delta(n)$ matters for the first auction equilibrium and that the auctioneer receives the entire competitive advantage in terms of increased revenue in the first auction.

PROPOSITION 3. (*Symmetric Bayesian Nash Equilibrium in the First Auction with Competitive Advantage*) *In model $\mathcal{M}_3$, the symmetric Bayesian Nash equilibrium in the first period is given by:*

$$\beta_{1i}(v) = v + \Delta(n) - \frac{1}{F(v)^{n-1}} \int_0^v F(x)^{n-1} dx, \qquad i = 1, \dots, n$$

Therefore, the auctioneer’s expected revenue in the first period is:

$$R_{31}(n) = R_{11}(n) + \Delta(n) = \int_0^1 n(n-1) x F(x)^{n-2} [1 - F(x)] f(x) dx + \Delta(n)$$

In particular, when the values of the bidders are uniformly distributed on $[0,1]$, the corresponding linear bid functions are:

$$\beta_{1i}(v) = \frac{n-1}{n} v + \Delta(n), \qquad i = 1, \dots, n$$

In addition, the expected revenue of the auctioneer in this first auction is:

$$R_{31}(n) = \frac{n-1}{n+1} + \Delta(n)$$

Therefore, the total revenue of the auctioneer with the mechanism (model) $\mathcal{M}_3$ is:

$$R_3(n) = R_{31}(n) + R_{32}(n)$$

Where $R_{32}(n)$ is the expected revenue of the auctioneer in the second period auction.

Proof: See Appendix.

Let $\beta_0(v)$ be the solution to the typical first-price auction. Then, we can write: $\beta_0(v) = v - \frac{1}{F(v)^{n-1}} \int_0^v F(x)^{n-1} dx$ and $\beta(v) = \beta_0(v) + \Delta(n)$.

Therefore, the competitor acts as if the value of the object in this auction was augmented by $\Delta(n)$. And this is, in fact, the case, since $\Delta(n)$ is indeed the additional expected net benefit accrued to his utility when he wins this first auction. Note that, given the expected benefit (in the second auction) of winning the first auction, the bid may exceed the players' value.

Hence, the asymmetry-creation rule increases the expected revenue of the auctioneer in the first auction. Conversely, since the mechanism gives a competitive advantage in the second auction to the first-auction winner, we may expect that the second auction will be less competitive, reducing the auctioneer's payoff in that auction.

Therefore, the biggest challenge for the mechanism designer is to design the competitive advantage in the second auction in such a way that the auctioneer's aggregated expected return in the two auctions increase.

In what follows, we analyze two possible implementations of the rule for creating asymmetry and compare the payoffs for the case of a uniform distribution. Both implementations are based on the following competitive advantage rule.

Definition 4. ***The Matching-price asymmetry-creation competitive-advantage rule.*** *The winner of the first auction can guarantee winning the second auction by matching its equilibrium price.*

Therefore, the asymmetry-creation rule allows the winner of the first auction to win the second auction after its equilibrium price has been determined, by matching that equilibrium price. Hence, regardless of the specific selling mechanism chosen for the second auction, that player has a dominant strategy, which is to bid $\beta_{2w}(v) = 0$ for any value v, and then, after the auction concludes, match the equilibrium price p if and only if $v \geq p$. But then, for the losers of the first auction, the second auction is equivalent to an auction with $n-1$ players and a secret reserve price. The secret reserve price is the value of the winner of the first auction, which distribution is F. The following corollary summarizes the dominant strategy in the second auction of a winner of the first auction under the above asymmetric creation rule.

Corollary 2. ***Dominant strategy of a winner of the first auction in the second auction under the matching price rule.*** *Suppose the Matching-price asymmetry-creation rule is applied to the second auction. Then, the winner of the first auction's dominant strategy is to bid zero in the bidding stage of the second auction and then match the equilibrium price if and only if her value is at least that equilibrium price.*

Proof: Trivial.

3.1. The second price selling mechanism for the second auction

Suppose, first, that the second object is sold using a second price auction. Name that model $\mathcal{M}_{31}$, the first implementation of the model $\mathcal{M}_3$.

Definition 4. *Model $\mathcal{M}_{31}$: two sequential sealed-bid auctions without a reserve price. The first auction is a first price sealed bid auction, the second auction is a second price sealed bid, and includes the Matching-price asymmetry-creation rule.*

We look for a Bayesian Nash Equilibrium (BNE) $\left(\beta_{21}(.), \ldots, \beta_{2n}(.)\right)$ where $\beta_{2i}(\cdot)$, $i = 1, \ldots, n$ are linear and non-decreasing functions from $[0,1]$ to $[0,1]$ satisfying $0 \leq \beta_{2i}(v_i) \leq v_i$, for all $v_i \in [0,1], i = 1, \ldots, n$.

Let 1 be the winner of the first auction. Recall that 1 has a dominating strategy to bid $\beta_{21}(v) = 0$, and then to match the highest bid of the other players, if his value is not lower than that highest bid.

$\beta_{21}(v) = 0$; if $v_1 \geq y = \max_{j>1} \beta_{2j}(v_j)$, then match bid y and win the object.

Also recall that the $n-1$ losers of the first auction play a $(n-1)$-player auction with a secret reserve price r, which distribution is $F(r)$. In the present implementation, they play a second-price auction where the secret reserve price is player 1's value. Hence, all losers of the first auction have a weakly dominating strategy, namely to bid their own values. Therefore, in equilibrium,

$$\beta_{2j}(v_j) = v_j, j = 2, \ldots n$$

The next proposition summarizes the second period equilibrium.

PROPOSITION 4. ***Bayesian Nash Equilibrium in the Second Auction with Competitive Advantage*****.** *In model $\mathcal{M}_{31}$ with a second-price second auction, the Bayesian Nash equilibrium in the second period auction, when player 1 is the winner of the first auction, and all losers of the first auction play identical strategies is given by:*

Player 1's strategy:

$$\beta_{21}(v_1) = 0$$

$$If\ v_1 \geq y = \max_{j>1} v_j \quad then\ match\ bid\ y\ and\ win\ the\ good.$$

Player j's strategy, j ≥ 2:

$$\beta_{2j}(v_j) = v_j$$

Proof: Immediate.

Note that, from the point of view of the auctioneer, the second auction is equivalent to a second-price auction with $n-1$ players instead of n players. Therefore, the competitive advantage mechanism induces a reduction in competition in the second auction. This is the cost to the auctioneer of designing a mechanism to induce asymmetry among the players. The following proposition presents the expected revenue of the auctioneer in the second auction, the expected (ex-ante) utilities in the second auction of the winner and of a loser of the first auction, and the corresponding competitive advantage.

PROPOSITION 5. ***Auctioneer's revenue in the Bayesian Nash Equilibrium in the Second Auction with Competitive Advantage*****.** *In the asymmetry-creating model $\mathcal{M}_{31}$ with a second-price second auction, the expected revenue of the seller in the second auction given the dominant strategy profile presented in Proposition 4 is:*

$$R_{32}(n) = 1 - (n-1)\int_0^1 F(y)^{n-2} dy + (n-2)\int_0^1 F(y)^{n-1} dy = R_{11}(n-1)$$

In the particular case of the uniform distribution, that expression reduces to:

$$R_{32}(n) = \frac{n-2}{n} = R_{11}(n-1)$$

The expected utility in the second auction of a winner of the first auction is:

$$u_w(n) = \int_0^1 \int_0^{v_1} (v_1 - y) f_2(y) dy\, f(v_1) dv_1$$

The expected utility in the second auction of a loser of the first auction is:

$$u_l(n) = \int_0^1 \int_0^{v_2} (v_2 - y)F(y)f_2(y)dy\, f(v_2)dv_2$$

Where $f_2(y) = (n-1)(n-2)f(y)F(y)^{n-3}\big(1-F(y)\big)$ *is the probability density function of the second highest value among the values of the* $n-1$ *losers.*

Furthermore, the competitive advantage is:

$$\Delta(n) = (n-1)\int_0^1 F(y)^{n-2}dy - (3n-5)\int_0^1 F(y)^{n-1}dy + \frac{(3n-1)(n-2)}{n}\int_0^1 F(y)^n dy - \frac{(n-1)(n-2)}{n}\int_0^1 F(y)^{n+1}dy$$

In particular, if the values of all players are uniformly distributed on $[0,1]$*, then:*

$$\Delta(n) = \frac{12}{n(n+1)(n+2)}$$

Proof: See Appendix.

In the first auction, the expected revenue of the auctioneer is the expected revenue of a standard first price auction with n-participants, plus the term $\Delta(n)$.

Therefore, $R_{31}(n) = R_{11}(n) + \Delta(n)$. Thus, the aggregated expected revenue of the auctioneer in both auctions is:

$$R_3(n) = R_{11}(n) + \Delta(n) + R_{11}(n-1)$$

Hence, the expected revenue of the auctioneer in the competitive advantage model is higher than the typical, no-reservation-price, first price auction if and only if:

$$R_3(n) = R_{11}(n) + \Delta(n) + R_{11}(n-1) > 2R_{11}(n) = R_1(n) \Leftrightarrow \Delta(n) > R_{11}(n) - R_{11}(n-1)$$

The following proposition explicates this comparison.

PROPOSITION 6. *Condition for the revenue dominance of the asymmetry-creation mechanism*. *In the asymmetry-creation model* $\mathcal{M}_{31}$ *with a second-price second auction and* $n \geq 3$ *players, the new mechanism yields a higher payoff to the auctioneer than the traditional first price auction with no reservation price if and only if:*

$$(2n^2 - 6n + 2)\int_0^1 F(y)^n dy > (n-3)n\int_0^1 F(y)^{n-1}dy + (n-1)(n-2)\int_0^1 F(y)^{n+1}dy$$

In particular, if the values of all players are uniformly distributed in $[0,1]$*, then that condition if satisfied only if there are exactly* $n = 3$ *players. If there are* 4 *participants, the expected revenue is the same and it is reduced if there are at least* 5 *participants.*

Proof. See Appendix.

3.2. The first price selling mechanism for the second auction

Suppose now that the second object is sold using a first price auction. Name that mechanism $\mathcal{M}_{32}$, the second implementation of model $\mathcal{M}_3$.

Definition 5. *Model $\mathcal{M}_{32}$: two sequential sealed-bid auctions without a reserve price. The first auction is a first price sealed bid auction, the second auction is a first price sealed bid, and includes the Matching-price asymmetry-creation rule.*

We look again for a Bayesian Nash Equilibrium (BNE) $\left(\beta_{21}(.), \beta_{22}(.), \beta_{2n}(.)\right)$ where $\beta_{2i}(\cdot)$, $i = 1, \dots, n$ are linear and non-decreasing functions from $[0,1]$ to $[0,1]$ satisfying $0 \leq \beta_{2i}(v_i) \leq v_i$, for all $v_i \in [0,1], i = 1, \dots, n$, where player 1 is the winner of the first auction and the other players are the losers of the first auction.

Recall that 1 has a dominating strategy to bid $\beta_{21}(v) = 0$, and then to match the highest bid of the other players, as long as his value is not lower than that highest bid.

$$\beta_{21}(v) = 0$$

$$If\ v_1 \geq y = \max_{j>1} \beta_{2j}(v_j),\ \ then\ match\ bid\ y\ and\ win\ the\ good.$$

Also recall that the $n-1$ losers of the first auction play a $(n-1)$-player first-price auction with a secret reserve price r, which distribution is $F(r)$. The secret reserve price is player 1's value.

The following proposition summarizes the second period equilibrium.

PROPOSITION 7. *Bayesian Nash Equilibrium in the Second Auction with Competitive Advantage*. *In model $\mathcal{M}_{31}$, the Bayesian Nash equilibrium in the second period first-price auction, when player 1 is the winner of the first auction, and all losers of the first auction play identical strategies is given implicitly by:*

Player 1's strategy:

$$\beta_{21}(v_1) = 0$$

$$If\ v_1 \geq y = \max_{j>1} v_j\ \ \ then\ match\ bid\ y\ and\ win\ the\ good.$$

Player j's strategy, j ≥ 2*:*

$$\beta_{2j}(v_j) = v_j - \frac{1}{F\left(\beta_{2j}(v_j)\right) F(v_j)^{n-2}} \int_0^{v_2} F\left(\beta_{2j}(v)\right) F(v)^{n-2} dv$$

In particular, when the values of the bidders are uniformly distributed on $[0,1]$*, i.e.,* $F(v) = v$*, the corresponding linear bid equilibrium is:*

$$\beta_{2j}(v_j) = \frac{n-1}{n} v_j \coloneqq \beta_2(v_j), \quad j = 2, \dots n \tag{1}$$

In that case, the expected utility of the winner of the first auction is:

$$u_w(n) = \frac{1}{2}\left[1 - \frac{1}{n^2}\frac{n-1}{n+1}(n-1)(n+3)\right]$$

And the expected utility of a loser j of the first auction is:

$$u_l(n) = \frac{1}{n^2}\frac{n-1}{n+1}$$

Therefore, the competitive advantage is:

$$\Delta(n) = \frac{1}{2}\frac{3n-1}{n^2(n+1)}$$

Proof: See Appendix.

Note that (1) is also the solution of a regular n-players auction without distinction among the players. Therefore, in the uniform distribution case, the $n-1$ losers in the first auction behave as if no one had competitive advantage over the others.

It is also noteworthy that regardless of who gets the object, whether the winner or a loser of the first auction, the auctioneer receives the highest of the bids. If $\beta_2(v_j)$ is a nondecreasing function, the auctioneer's expected revenue in the second auction is:

$$R_{32}(n) = E\left[\max_{j\geq 2} \beta_2(v_j)\right] = E\left[\beta_2\left(\max_{j\geq 2} v_j\right)\right] = \int_0^1 \beta_2(x) dF(x)^{n-1}$$

$$R_{32}(n) = (n-1)\int_0^1 \beta_2(x)F(x)^{n-2}f(x)dx = \left(\frac{n-1}{n}\right)^2$$

Comparing the expected revenue in the second period of the basic first-price auction we have:

$$R_{12}(n) = \frac{n}{n+1} > \left(\frac{n-1}{n}\right)^2 = R_{32}(n)$$

Therefore,

$R_{22}(n) > R_{12}(n) > R_{32}(n)$:

Thus, the competitive advantage model yields the lowest expected revenue in the second auction among the three formats studied here. This is precisely what makes winning in the first auction more attractive to the players, which yields higher bids in that first auction. It is the cost the auctioneer has to bear in order to create the endogenous asymmetry. It will become clear in this paper that it is worthwhile bearing that cost, as the aggregated, two-period expected revenue is the highest when this model is used.

Note that:

$$\frac{u_w(n)}{u_l(n)} = \frac{1}{2}\frac{1}{n-1}[5n-3] = \frac{5n-3}{2n-2} \Rightarrow \lim_{n\to\infty}\frac{EU_1(n)}{EU_2(n)} = \frac{5}{2}.$$

Hence, the victorious player in the first auction maintains a utility that is over twice the one of a looser in the second auction as the number of players increases. This shows that the strategic advantage remains important, even when competition expands.

The following proposition shows that if the values are uniformly distributed in [0,1], then the model with the competitive advantage rule and the first price selling mechanism for the second auction yields a higher payoff for the seller than the traditional first price auction without reservation price for any number of players ($n \geq 2$) and that it yields a higher payoff than the traditional first price auction with optimal reservation price if there are at least $n = 4$ players.

PROPOSITION 8. *Superiority of the Asymmetry-creation mechanism in the uniform distribution case*. *In the asymmetry-creating model $\mathcal{M}_{32}$ with a first-price second auction and identical and independent uniform distributions for the players' values, the new mechanism yields a higher payoff to the auctioneer than the traditional first price auction with no reservation price. Furthermore, it yields a higher payoff than the traditional first price auction with optimally chosen reservation price if there are at least $n = 4$ players.*

Proof: See Appendix.

3.3. Discussion on the advantages of the mechanism that generates asymmetry vis-à-vis the traditional mechanisms

Proposition 8 has shown that the model with competitive advantage and first price second auction yields a higher payoff than the traditional models, both with or without reservation price, if there are at least four players. Therefore, by strategically using the recurrent characteristics of certain auctions, the auctioneer may increase its expected payoff. This revenue increasing property of the proposed model makes it quite attractive for all sorts of applications, particularly those involving recurrent public purchases. Table 1 presents the corresponding revenues when there are 3 to 15 players.

Table 1. Auctioneer's expected revenue comparison for three possible sequential auction formats: traditional second price model ($\boldsymbol{R_1}$), second

price model with optimally chosen reserve price (R_2), and the competitive advantage model (R_{32}), for number of participants n from 3 to 15.

n	$R_1(n)$	$R_2(n)$	$R_{32}(n)$	$\frac{R_{32}(n)-R_1(n)}{R_1(n)}$	$\frac{R_{32}(n)-R_2(n)}{R_2(n)}$	$\frac{R_2(n)-R_1(n)}{R_1(n)}$
3	1.00000	1.06250	1.05556	0.05556	-0.00654	0.0625000
4	1.20000	1.22500	1.23125	0.02604	0.00510	0.0208333
5	1.33333	1.34375	1.35333	0.01500	0.00713	0.0078125
6	1.42857	1.43304	1.44246	0.00972	0.00658	0.0031250
7	1.50000	1.50195	1.51020	0.00680	0.00549	0.0013021
8	1.55556	1.55642	1.56337	0.00502	0.00446	0.0005580
9	1.60000	1.60039	1.60617	0.00386	0.00361	0.0002441
10	1.63636	1.63654	1.64136	0.00306	0.00295	0.0001085
11	1.66667	1.66675	1.67080	0.00248	0.00243	0.0000488
12	1.69231	1.69235	1.69578	0.00205	0.00203	0.0000222
13	1.71429	1.71430	1.71724	0.00173	0.00172	0.0000102
14	1.73333	1.73334	1.73588	0.00147	0.00147	0.0000047
15	1.75000	1.75000	1.75222	0.00127	0.00127	0.0000022

The first column is the number of participants. Columns 2, 3 and 4 present the corresponding expected seller's revenue for the traditional first or second price model without reservation price (column 2), first price auction with optimally chosen reservation price (column 3) and new competitive advantage model (column 4). The next columns compare the expected revenue of the three models: $\mathcal{M}_4$ with $\mathcal{M}_1$ (column 5), $\mathcal{M}_4$ with $\mathcal{M}_2$ (column 6) and $\mathcal{M}_2$ with $\mathcal{M}_1$ (column 7).
Source: Authors' calculations.

The table shows the significant revenue advantage of the new model. When there are 4 players the competitive advantage design yields an increase of 2.6% in the auctioneer's revenue while preserving the property of always selling the object.

Note that the well know result regarding the revenue-dominance of the reserve price auction design in comparison with the traditional model without reservation price, is significant only when there are very few participants. With 7 participants that gain reduces to about 0.1 percentage point (column 7), in line with Klemperer (2002), which asserts the importance of competition in auction design. In comparison, the competitive advantage model yields a fivefold percentage benefit in the case of seven players (column 5).

Furthermore, it is worth noting that, according to Engelbrecht-Wiggans (1987), when there is an endogenous decision to participate in the auction and a sunk participation cost, including a reserve price can reduce the number of participants, thereby reducing competition which, in turn, reduces the auctioneer's return. From this perspective, the comparison should be made precisely with the model without a strategic reserve price, model $\mathcal{M}_1$, which generates a clearly inferior return compared to the proposed model, $\mathcal{M}_{32}$ even when only three players compete.

Most importantly, the item is always sold in the proposed auction format, even though possibly at lower prices in the second auction. That makes this mechanism effective, unlike the one that includes a strategically chosen reserve price. Note, however, that just like in the mechanism with optimally chosen reserve price, the proposed mechanism is not efficient, albeit for different reasons. While in the traditional mechanism, inefficiency results arise from the possibility of the item not being sold when it should be (Myerson 1981; Riley & Samuelson, 1981), in the proposed mechanism, the item is always sold, but in the second auction, it might be sold to a participant (winner of the first auction) who values it less than her competitor (a loser of the first auction).

The fact that the proposed mechanism always sells the objects is especially relevant in applications to the public sector, where a failed auction, one in which the item is not sold, brings about a wide range of additional costs, from the cost of organizing another bidding process to reputational costs, for example (Casady et al., 2023). Faced with these costs, the literature even finds situations where the public manager may distort his behavior to favor cartel formation (Tanaka & Hayashi, 2016). This type of concern disappears when using the competitive advantage mechanism.

Another advantage, especially relevant for the public sector, is that in the real world the public manager does not have discretion in the strategic choice of the reserve price to be used in the auction. In fact, this price is typically determined based on accounting studies that seek to assess the value of the good being sold (a company being privatized, for example, or a good being procured). Now, Auction Theory convincingly shows that, at least in the context of independent private values[5], the optimal reserve price is strictly above the item's value for the auctioneer (Myerson, 1981; Riley & Samuelson, 1981), and this difference can be substantial. For example, in the parameterization studied in this article, the item's value for the auctioneer was zero while the reserve price was ½, which is the expected value that any participant assigns to the item. By replacing the discussion regarding the reserve price with a discussion about the competitive advantage, the public manager solves the nightmare of auction failure.

Furthermore, the literature is not consensual with respect to the use and properties of the optimal reserve price when the number of participants varies. Menicucci

[5] When there is correlated information, as in the case of the common value model or the affiliated private value model, Levin & Smith (1996) show that the seller's optimal reservation price converges to his true value, as the number of bidders increases.

(2021) shows that when the virtual valuation function (Myerson 1981) is not monotonic, the optimal reserve price weakly increases with the number of participants. Conversely, when participation is endogenous and costly, Engelbrecht-Wiggans (1987) shows that if potential participants only learn the value of the object when they incur the participation cost, then a null reservation price may yield higher expected returns to the auctioneer, because positive reserve prices reduce participation. These results reinforce support for an alternative model to the strategically chosen reserve price model.

Additionally, by creating a competitive advantage for the winner of an auction, the mechanism establishes a dynamic link between this player and the auctioneer, causing the player to have an interest in behaving appropriately to maintain the privilege guaranteed by the victory until the next auction. Suppose, for example, that what is being auctioned is a public-private partnership (PPP). The literature on the subject finds, both theoretically and empirically (Laffont 2005, p.245; Bugarin & Ribeiro 2021; Estache & Quesada, 2001; Gagnepain et al. 2013; Guasch et al. 2006, 2007, 2008), ample evidence of non-compliance with contractual commitments throughout the concession, leading to costly renegotiation processes. The competitive advantage mechanism can become another incentive for compliance with commitments, by including the possibility of losing the competitive advantage in the next PPP auction if the winner fails to fulfill the contractual commitments in the current partnership.

Finally, the competitive advantage also has the potential to reduce the chances of adventurers who do not intend to fulfill contractual commitments from winning the bidding processes, both in the first auction, as they know they will lose the advantage in the second if they win and fail to meet the conditions, and in the second auction, as the serious winner will have an advantage over them.

4. CONCLUSIONS

This article begins by observing that many economic transactions conducted through auctions are recurring, such as the purchase of office supplies for public services, the hiring of maintenance services for public parks, and the organization of annual fairs.

Building on this idea, the present research aims to leverage the repeated interactions between the auctioneer and the participants to create a mechanism that enhances competition and increases the auctioneer’s expected revenue. The proposed mechanism involves a novel approach to repeated auctions: the first auction is conducted as a standard first-price auction. However, in the second auction, a modification is introduced that deviates from the standard format. This modification, known as the "asymmetry creation rule" or the "competitive advantage mechanism," allows the winner of the first auction to match the equilibrium bid in the second auction, thereby securing the item.

The article demonstrates that in the context of symmetric independent private values, identically distributed according to the uniform distribution, the competitive advantage mechanism generates an expected return well above that of simply applying two standard consecutive auctions without a reserve price. If we compare to the auction using the optimal reserve price that maximizes the auctioneer's expected return, the mechanism generates higher expected revenue than this mechanism if there are at least 4 participants, without incurring the risk of auction failure (not selling the object).

Although it does not reach the auctioneer's expected revenue when the optimal reserve price is used, if there are only two or three participants, the competitive advantage mechanism has several properties that make it particularly desirable in practical applications, especially in the public sector.

One of the most significant benefits is that it ensures that there will always be sales of the items, which cannot be ensured in the case of using a reserve price. This property is especially relevant when considering the different organizational and reputational costs associated with a "failed auction”, where the item is not sold.

In real-world implementations in the public sector, reserve prices are typically determined by an accounting assessment of the auctioned item's value. This estimate of the item's value for the auctioneer is unlikely to coincide with the optimal value that maximizes the auctioneer's revenue. Indeed, Myerson (1981) and Riley & Samuelson (1981) show that the optimal reserve price is higher than the auctioneer's reservation value in the risk-neutral independent-private-values (IPV) auction model that we study here. Thus, the competitive advantage mechanism may, in practice, generate expected revenue for the auctioneer higher than the mechanism with a reserve price equal to the item's value for the auctioneer for any number of participants.

Finally, by creating a dynamic bond between the auctioneer and the winner, the proposed mechanism incentivizes the winner to fulfill all commitments made in the first auction to avoid losing the strategic advantage in the subsequent auction. This provides greater control by the public sector over the auction winner, a recurring concern in mechanism design applied to the public sector.

Naturally, there are questions that can be raised regarding the implementation of the proposed mechanism.

One concern is that the mechanism violates the principle of equality, as one participant has an advantage over the others in the second auction. This characteristic suggests that some participants (losers in the first auction) may challenge the asymmetry creation rule in the second auction in Court. For this reason, great effort must be put into convincing society, and especially the Judiciary, of the advantages of the mechanism before implementation.

Furthermore, and associated with the above questioning, although the proposed mechanism ensures that the item will always be sold, it does not guarantee that the item will be sold to the participant who values it the most, i.e., the mechanism is not efficient, just like the mechanism with the optimal reserve price, albeit for different reasons.

This article makes an original contribution to the literature on repeated auctions by shifting the focus from issues related to discouraging collusion among participants to the search for practical alternative mechanisms that can expand the auctioneer's expected revenue. This research can be extended in multiple directions, some of which are presented below as suggestions for future research.

The model presented here assumes that the auctioneer assigns zero value to the item. Therefore, the most natural comparison is indeed with the traditional model with a zero reserve price, a comparison that is strongly favorable to the proposed model. Considering that in the public sector, the reserve value is the value of the item for the auctioneer, the proposed model should also be compared with an implementation including a reserve value identical to the value of the item for the auctioneer, and the two possible implementations should be compared.

Given that sequential public tenders usually take significant periods of time to complete, intertemporal discounting between the two auctions is another path for extension. It is also necessary to understand how the results found here are affected by the number of participants when this number is endogenous. Additionally, a model with more than two consecutive auctions or where the objects to be sold have some degree of complementarity or substitutability should be considered.

Additionally, the model should be extended to more general probability distributions of participants' values in the symmetric independent private values model, and its applicability to situations where values are possibly affiliated should be analyzed.

Finally, this research has shown that in the competitive advantage model designing the second auction as a second price or as a first price yield different expected revenues to the auctioneer. Therefore, future research should explore alternative forms for the second and possibly the first auction.

REFERENCES

Abreu, D. (1988). On the theory of infinitely repeated games with discounting. *Econometrica: Journal of the Econometric Society*, 383-396.

Aoyagi, M. (2003). Bid rotation and collusion in repeated auctions. *Journal of Economic Theory*, *112*(1), 79-105.

Ausubel, L. M., & Cramton, P. (2004). Vickrey auctions with reserve pricing. *Economic Theory*, *23*(3), 493-505.

Bugarin, M., & Ribeiro, F. (2001). The paradox of concessions in developing countries. *Brazilian Review of Econometrics* 41(1): 69-100. https://doi.org/10.12660/bre.v41n12021.83397.

Casady, C. B., Petersen, O. H., & Brogaard, L. (2023). Public procurement failure: The role of transaction costs and government capacity in procurement cancellations. *Public Management Review*, 1–28. https://doi.org/10.1080/14719037.2023.2231945.

De Silva, D. G., Jeitschko, T. D., & Kosmopoulou, G. (2005). Stochastic synergies in sequential auctions. *International Journal of Industrial Organization*, *23*(3-4), 183-201.

Elyakime, B., Laffont, J. J., Loisel, P., & Vuong, Q. (1994). First-price sealed-bid auctions with secret reservation prices. *Annales d'Economie et de Statistique*, 115-141.

Engelbrecht-Wiggans, R. (1987). On optimal reservation prices in auctions. *Management Science*, *33*(6), 763-770.

Estache, A., & Quesada, L. (2001). Concession contract renegotiations: Some efficiency versus equity dilemmas (Policy Research Paper No. 2705). Washington, DC: The World Bank. https://openknowledge.worldbank .org/handle/10986/19495

Fudenberg, D., & Maskin, E. (1986) The folk theorem in repeated games with discounting and with incomplete information', *Econometrica*, *54*: 533-554.

Gagnepain, P., Ivaldi, M., & Martimort, D. (2013). The cost of contract renegotiation: Evidence from the local public sector. American Economic Review, *103*(6), 2352–1383. http://dx.doi.org/10.1257/aer.103.6.2352.

Guasch, J. L., Laffont, J.-J., & Straub, S. (2006). Renegotiation of concession contracts: A theoretical approach. Review of Industrial Organization, *29*, 55–73. http://dx.doi.org/10.1007/s11151-006-9109-5.

Guasch, J. L., Laffont, J.-J., & Straub, S. (2007). Concessions of infrastructure in Latin America: Government-led renegotiations. Journal of Applied Econometrics, *22*(7), 1267–1294. http://dx.doi.org/10.1002/jae.987.

Guasch, J. L., Laffont, J.-J., & Straub, S. (2008). Renegotiation of concession contracts in Latin America: Evidence from the water and transport sectors. International Journal of Industrial Organization, *26*(2), 421–442. http://dx.doi.org/10.1016/j.ijindorg.2007.05.003.

Jofre-Bonet, M., & Pesendorfer, M. (2014). Optimal sequential auctions. International Journal of Industrial Organization, 33, 61-71.

Kaplan, T. R., & Zamir, S. (2012). Asymmetric first-price auctions with uniform distributions: analytic solutions to the general case. *Economic Theory*, *50*(2), 269-302.

Klemperer, P. (2002). What really matters in auction design. Journal of Economic Perspectives, *16*(1), 169-189.

Kirkegaard, R. (2022). Efficiency in asymmetric auctions with endogenous reserve prices. *Games and Economic Behavior*, *132*, 234-239.

Krishna, V. (2002). Auction theory. *San Diego: Academic Press*.

Laffont, J.-J. (2005). Regulation and development. Cambridge: Cambridge University Press.

Levin, D., & Smith, J. L. (1996). Optimal reservation prices in auctions. *The Economic Journal*, *106*(438), 1271-1283.

Marshall, R. C., & Marx, L. M. (2012). The economics of collusion: Cartels and bidding rings. MIT Press.

McAfee, R. P., & McMillan, J. (1992). Bidding rings. *The American Economic Review*, 579-599.

McHugh, R. (2020) How Much Paper Schools Use & How Much It Costs Them. *Record Nations,* Jan. 14, 2000. https://www.recordnations.com/blog/how-much-paper-schools-use-how-much-costs/

Menezes, F. M., & Monteiro, P. K. (2004). *An introduction to auction theory*. OUP Oxford.

Menicucci, D. (2021). In the basic auction model, the optimal reserve price may depend on the number of bidders. *Journal of Economic Theory*, *198*, 105371.

Milgrom, P., & Weber, R. J. (1982). The value of information in a sealed-bid auction. *Journal of Mathematical Economics*, *10*(1), 105-114.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of operations research*, *6*(1), 58-73.

Naser, A.Y., Alwafi, H., Al-Daghastani, T. *et al.* Drugs utilization profile in England and Wales in the past 15 years: a secular trend analysis. *BMC Prim. Care*, *23*, 239 (2022). https://doi.org/10.1186/s12875-022-01853-1

Phillips, O. R., Menkhaus, D. J., & Coatney, K. T. (2003). Collusive practices in repeated English auctions: Experimental evidence on bidding rings. *American Economic Review*, *93*(3), 965-979.

Riley, J. G., & Samuelson, W. F. (1981). Optimal auctions. *American Economic Review*, *71*(3), 381-392.

Salant, D. J., & Cabral, L. (2019). Sequential auctions and auction revenue. *Economics letters*, *176*, 1-4.

Skrzypacz, A., & Hopenhayn, H. (2004). Tacit collusion in repeated auctions. *Journal of Economic Theory*, *114*(1), 153-169.

Sorin, S. (1992). Repeated games with complete information. *Handbook of game theory with economic applications*, *1*: 71-107.

Tanaka, S., & Hayashi, S. (2016), Collusion between public procurers and suppliers in the context of Japan's public procurement: The role of the risks of "unsuccessful procurement", *Journal of Public Procurement*, Vol. 16 No. 3, pp. 291-311. https://doi.org/10.1108/JOPP-16-03-2016-B003.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, *16*(1), 8-37.

Wilson, R. (1966). *Competitive bidding with disparate information.* Working paper No. 114. Graduate School of Business, Stanford University.

Wilson, R. (1967). Competitive bidding with asymmetric information. *Management Science*, *13*(11), 816-820.

Xu, H., & Cavallo, R. (2022). The Strange Role of Information Asymmetry in Auctions—Does More Accurate Value Estimation Benefit a Bidder? *Proceedings of the AAAI Conference on Artificial Intelligence 36*(5), pp. 5236-5243.

APPENDIX

PROOF OF PROPOSITION 3.

We look for a symmetric, differentiable, strictly increasing equilibrium $\beta(v)$.

The expected utility of a bidder of value v who bids b is:

$$EU(b;v) = \left(v - b + u_w(n)\right)F\left(\beta^{-1}(b)\right)^{n-1} + u_l(n)\left(1 - F\left(\beta^{-1}(b)\right)^{n-1}\right)$$

$$EU(b;v) = \left(v - b + \Delta(n)\right)F\left(\beta^{-1}(b)\right)^{n-1} + u_l(n)$$

The corresponding FOC is:

$$-F\left(\beta^{-1}(b)\right)^{n-1} + (n-1)\left(v - b + \Delta(n)\right)F\left(\beta^{-1}(b)\right)^{n-2}f\left(\beta^{-1}(b)\right)[(\beta^{-1})'(b)] = 0$$

In equilibrium, $b = \beta(v) \Rightarrow v = \beta^{-1}(b)$. Furthermore, $(\beta^{-1})'(b) = \left(\beta'(v)\right)^{-1}$.

Thus, the FOC rewrites as:

$$\left(v + \Delta(n)\right)(n-1)f(v)F(v)^{n-2} = \beta(v)(n-1)f(v)F(v)^{n-2} + \beta'(v)F(v)^{n-1}$$

Therefore, by the Fundamental Theorem of Calculus,

$$\beta(v)F(v)^{n-1} = \int_0^v \left(x + \Delta(n)\right)(n-1)f(x)F(x)^{n-2}dx$$

Hence the solution:

$$\beta(v) = \frac{1}{F(v)^{n-1}}\int_0^v \left(x + \Delta(n)\right)(n-1)f(x)F(x)^{n-2}dx$$

$$= \frac{1}{F(v)^{n-1}}\int_0^v x(n-1)f(x)F(x)^{n-2}dx + \frac{\Delta(n)}{F(v)^{n-1}}\int_0^v (n-1)f(x)F(x)^{n-2}dx$$

$$\beta(v) = v + \Delta(n) - \frac{1}{F(v)^{n-1}}\int_0^v F(x)^{n-1}dx$$

Recall that the distribution of the maximum of value the n competitors is $F(v)^n$ and its density is $dF(v)^n = nf(v)F(v)^{n-1}dv$. Therefore, the seller's expected revenue is:

$$R_{31}(n) = \int_0^1 \beta(v)dF(v)^n = n\int_0^1 \left[v - \frac{1}{F(v)^{n-1}}\int_0^v F(x)^{n-1}dx\right]f(v)F(v)^{n-1}dv + \Delta(n)$$

$$= n\int_0^1 \beta_0(v)f(v)F(v)^{n-1}dv + \Delta(n) = R_{11}(n) + \Delta(n)$$

Where $\beta_0(v) = v - \frac{1}{F(v)^{n-1}}\int_0^v F(x)^{n-1}dx$ is the solution to the standard first price auction without reservation price.

In the specific uniform distribution case, we have $F(x) = x, f(x) = 1$ and the equilibrium becomes:

$$\beta(v) = \frac{1}{v^{n-1}} \int_0^v \big(x + \Delta(n)\big)(n-1)x^{n-2}dx = \frac{n-1}{n} v + \Delta(n)$$

Since $R_{11}(n) = \frac{n-1}{n+1}$, it follows that $R_{31}(n) = \frac{n-1}{n+1} + \Delta(n)$.

PROOF OF PROPOSITION 5.

Let Y_2 be the second highest value among all losers in the second auction. Then, the distribution of Y_2 is[6]: $F_2(y) = F(y)^{n-1} + (n-1)\big(1 - F(y)\big)F(y)^{n-2}$

The corresponding density is: $f_2(y) = (n-1)(n-2)f(y)F(y)^{n-3}\big(1 - F(y)\big)$

Therefore, the expected revenue of the second period auction to the auctioneer is:

$$R_{32}(n) = \int_0^1 y f_2(y)dy = (n-1)(n-2)\left[\int_0^1 y f(y)F(y)^{n-3}dy - \int_0^1 y f(y)F(y)^{n-2}dy\right]$$

$$R_{32}(n) = 1 - (n-1)\int_0^1 F(y)^{n-2}dy + (n-2)\int_0^1 F(y)^{n-1}dy = R_{11}(n-1)$$

In the particular case of the uniform distribution, that expression reduces to:

$$R_{32}(n) = \frac{n-2}{n} = R_{11}(n-1)$$

Player 1, winner of the first auction, wins if his value is higher than the second highest value of the losers.

Therefore, 1's expected utility in the second auction is:

$$u_w(n) = \int_0^1 \int_0^{v_1} (v_1 - y)f_2(y)dy\, f(v_1)dv_1$$

Similarly, Player 2, a loser of the first auction, wins if:

(i) His value is the highest value of all the losers' values.

(ii) The second highest value among the losers is higher than the value of the winner of the first auction.

Therefore, 2's expected utility in the second auction is:

[6] The first summand is the probability that all $n-1$ values are smaller than y, whereas the second summand is the probability that one of the values is higher than y, and the other ones are smaller.

$$u_2(n) = \int_0^1 \int_0^{v_2} (v_2 - y)F(y)f_2(y)dy\, f(v_2)dv_2$$

The additional factor $F(y)$ in the above expression expresses the fact that 2 only wins when, in addition from having the highest value realization among all losers, the value realization of the winner is lower than the second highest value among the losers.

Therefore, the expected advantage in the second auction of winning the first, due to the competitive advantage mechanism, is:

$$\Delta(n) = u_1(n) - u_2(n) = \int_0^1 \int_0^{v} (v - y)(1 - F(y))f_2(y)dy\, f(v)dv$$

$$\Delta(n) = (n-1)\int_0^1 F(y)^{n-2}dy - (3n-5)\int_0^1 F(y)^{n-1}dy + \frac{(3n-1)(n-2)}{n}\int_0^1 F(y)^{n}dy - \frac{(n-1)(n-2)}{n}\int_0^1 F(y)^{n+1}dy$$

In the uniform distribution case:

$$\Delta(n) = 1 - \frac{3n-5}{n} + \frac{(3n-1)(n-2)}{n(n+1)} - \frac{(n-1)(n-2)}{n(n+2)} = \frac{12}{n(n+1)(n+2)}$$

PROOF OF PROPOSITION 6.

In the first auction, the expected revenue of the auctioneer is the expected revenue of standard first-price (or second-price) auction with n-participants, plus the term $\Delta(n)$.

Therefore, the expected revenue of the auctioneer in both auctions is:

$$R_4(n) = R_{11}(n) + \Delta(n) + R_{11}(n-1)$$

Therefore, the expected revenue of the auctioneer in the competitive advantage model is higher than the typical, no-reservation-price, first-price auction if and only if:

$$R_4(n) = R_{11}(n) + \Delta(n) + R_{11}(n-1) > 2R_{11}(n) = R_1(n) \Leftrightarrow \Delta(n) > R_{11}(n) - R_{11}(n-1)$$

This is equivalent to:

$$(n-1)\int_0^1 F(y)^{n-2}dy - (3n-5)\int_0^1 F(y)^{n-1}dy + \frac{(3n-1)(n-2)}{n}\int_0^1 F(y)^{n}dy - \frac{(n-1)(n-2)}{n}\int_0^1 F(y)^{n+1}dy >$$

$$1 - n\int_0^1 F(y)^{n-1}dy + (n-1)\int_0^1 F(y)^{n}dy$$

$$-\left(1-(n-1)\int_0^1 F(y)^{n-2}dy+(n-2)\int_0^1 F(y)^{n-1}dy\right)$$

$$\Leftrightarrow (2n^2-6n+2)\int_0^1 F(y)^n dy > (n-3)n\int_0^1 F(y)^{n-1}dy+(n-1)(n-2)\int_0^1 F(y)^{n+1}dy$$

For the uniform case,

$$\frac{2n^2-6n+2}{n+1} > (n-3)+\frac{(n-1)(n-2)}{n+2} \Leftrightarrow n < 4.$$

PROOF OF PROPOSITION 7.

Let 1 be the winner of the first auction and let 2 be a generic loser of the first auction.

Then, it is easy to see that 1 has a dominant strategy:

Bid $\beta_1(v_1)=0$, and, if the winning bid b_v is smaller than v_1, match that bid and win the object.

Suppose 2 bids b_2.

Then 2 wins the object if the following two conditions are satisfied:

(i) b_2 is greater than the highest bid made by the other losers:

$$b_2 > \beta_j(v_j), j=3,4,\dots,n \Leftrightarrow v_j < \beta_j^{-1}(b_2)$$

(ii) b_2 is higher than v_1:

$$b_2 > v_1 \Leftrightarrow v_1 < b_2$$

Condition (i) occurs with probability $\prod_{j=3}^n F(\beta_j^{-1}(b_2))$.

Condition (ii) occurs with probability $F(b_2)$

Therefore, the expected utility of 2 when her value is v_2 and she bids b_2 is:

$$EU_2(b_2;v_2)=(v_2-b_2)F(b_2)\prod_{j=3}^n F\left(\beta_j^{-1}(b_2)\right)$$

To easy the calculation of the first order condition, let us take the logarithm:

$$\ln\left(EU_2(b_2;v_2)\right)=\ln(v_2-b_2)+\ln\left(F(b_2)\right)+\sum_{j=3}^n \ln\left(F(\beta_j^{-1}(b_2)\right)$$

The corresponding first order condition is:

$$-\frac{1}{v_2-b_2}+\frac{f(b_2)}{F(b_2)}+\sum_{j=3}^{n}\frac{f\left(\beta_j^{-1}(b_2)\right)}{F\left(\beta_j^{-1}(b_2)\right)}\times\frac{1}{\beta_j'\left(\beta_j^{-1}(b_2)\right)}=0$$

We look for a "symmetric" equilibrium where all losers in the first auction follow the same strategy $\beta_j \equiv \beta$, $b_2 = \beta_j(v_2) = \beta(v_2)$ for $j = 2,3,\cdots,n$. Since $\beta^{-1}(b_2) = v_2$,

$$-\frac{1}{v_2-\beta(v_2)}+\frac{f\big(\beta(v_2)\big)}{F\big(\beta(v_2)\big)}+\sum_{j=3}^{n}\frac{f(v_2)}{F(v_2)}\times\frac{1}{\beta'(v_2)}$$
$$=-\frac{1}{v_2-\beta(v_2)}+\frac{f\big(\beta(v_2)\big)}{F\big(\beta(v_2)\big)}+(n-2)\frac{f(v_2)}{F(v_2)}\times\frac{1}{\beta'(v_2)}=0$$

The FOC may be written as:

$$\big(v_2-\beta(v_2)\big)\left[f\big(\beta(v_2)\big)\beta'(v_2)F(v_2)^{n-2}+(n-2)f(v_2)F\big(\beta(v_2)\big)F(v_2)^{n-3}\right]$$
$$+\big(1-\beta'(v_2)\big)F\big(\beta(v_2)\big)F(v_2)^{n-2}=F\big(\beta(v_2)\big)F(v_2)^{n-2}$$

The left-hand side of above equation is:

$$\frac{d}{dv_2}\left[\big(v_2-\beta(v_2)\big)F\big(\beta(v_2)\big)F(v_2)^{n-2}\right]$$

Therefore, from the Fundamental Theorem of Calculus,

$$\big(v_2-\beta(v_2)\big)F\big(\beta(v_2)\big)F(v_2)^{n-2}=\int_0^{v_2}F\big(\beta(v)\big)F(v)^{n-2}dv$$

Hence, for $v_2 > 0$,

$$v_2-\beta(v_2)=\frac{1}{F\big(\beta(v_2)\big)F(v_2)^{n-2}}\int_0^{v_2}F\big(\beta(v)\big)F(v)^{n-2}dv$$

Thus, the symmetric strategy $\beta(v_2)$ in the second auction of the players who lost the first auction is the implicit solution of the following equation:

$$\beta(v_2)=v_2-\frac{1}{F\big(\beta(v_2)\big)F(v_2)^{n-2}}\int_0^{v_2}F\big(\beta(v)\big)F(v)^{n-2}dv$$

Note that in this equilibrium the losers play as if they are the only participants in the second auction with a secret reservation price whose distribution is the same as their own valuation distribution. See, for example, Elyakime et al. (1994).

For the uniform distribution case, substituting $v = v_2$, $f(x) = 1$ and $F(x) = x$ in the FOC yields:

$$\frac{1}{\beta(v)}+\frac{(n-2)}{v\,\beta'(v)}=\frac{1}{v-\beta(v)}$$

$$\Rightarrow v\,\beta'(v)\left(v-\beta(v)\right)+(n-2)\beta(v)\left(v-\beta(v)\right)=v\beta(v)\beta'(v)$$

Let us find a linear solution $\beta(v)=Av+B$, then:

$$Av(v-Av-B)+(n-2)(Av+B)(v-Av-B)=v(Av+B)A$$

That is a polynomial equation that must hold for every value of v. Therefore, making $v\rightarrow 0$ it results:

$$(n-2)(-B^2)=0\Rightarrow B=0$$

Substituting in the previous equation:

$$A(1-A)v^2+(n-2)A(1-A)v^2=A^2v^2\Rightarrow(1-A)+(n-2)(1-A)=A$$

$$\Rightarrow A=\frac{n-1}{n}$$

Hence, the equilibrium bid for a loser of the first auction is:

$$\beta(v)=\frac{n-1}{n}v$$

Let us know calculate the expected utilities in the second auction of the winner and of the losers of the first auction.

Player 1 will win if $v_1>\beta(v_j), j=2,\dots,n\Leftrightarrow v_1>\frac{n-1}{n}v_j, j=2,\dots,n\Leftrightarrow v_j<\frac{n}{n-1}v_1$.

In particular, if $v_1>\frac{n-1}{n}$, he will win for sure.

Let $G(x)=F(x)^{n-1}$ be the distribution of the maximum of the realizations of the values of the other $n-1$ players. Then, $dG(x)=(n-1)f(x)F(x)^{n-2}dx=(n-1)x^{n-2}dx$.

Therefore, player 1's expected utility is:

$$EU_1(n)=\int_0^{\frac{n-1}{n}}\int_0^{\frac{n}{n-1}v_1}(v_1-\frac{n-1}{n}x)dG(x)\,dv_1+\int_{\frac{n-1}{n}}^1\int_0^1(v_1-\frac{n-1}{n}x)dG(x)\,dv_1$$

$$u_w(n)=\left[\frac{(n-1)^2}{n^3}\left[\frac{1}{n+1}-\frac{1}{2}(n+2)\right]+\frac{1}{2}\right]=\frac{1}{2}\left[1-\frac{1}{n^2}\frac{n-1}{n+1}(n-1)(n+3)\right]$$

Similarly, player 2 wins if $v_1<\frac{n-1}{n}v_2$ and $v_2>v_j, j=3,\dots,n$.

Let $H(x)=F(x)^{n-2}$ be the distribution of the maximum of the realizations of the values of the other $n-2$ players who lost the first auction. Then, the probability that 2 wins is: $F\left(\frac{n-1}{n}v_2\right)H(v_2)=F\left(\frac{n-1}{n}v_2\right)F(v_2)^{n-2}=\frac{n-1}{n}v_2^{n-1}$.

When 2 wins his utility is: $v_2-\frac{n-1}{n}v_2=\frac{1}{n}v_2$. Therefore, player 2's expected utility is:

$$u_l(n)=\int_0^1 \frac{1}{n}v_2\frac{n-1}{n}v_2^{n-1}dv_2=\frac{n-1}{n^2}\int_0^1 v_2^n\,dv_2=\frac{1}{n^2}\frac{n-1}{n+1}$$

Hence,

$$\Delta(n)=\frac{1}{2}\left[1-\frac{1}{n^2}\frac{n-1}{n+1}(n-1)(n+3)\right]-\frac{1}{n^2}\frac{n-1}{n+1}=\frac{1}{2}\frac{3n-1}{n^2(n+1)}$$

PROOF OF PROPOSITION 8.

Note that in the second auction the object is always sold for the highest bid among the $n-1$ losers in the first auction. Therefore, the expected revenue of the seller in this second auction is:

$$R_{32}(n)=\int_0^1 \beta(x)dF(x)^{n-1}=\int_0^1\frac{n-1}{n}x(n-1)x^{n-2}dx=\left(\frac{n-1}{n}\right)^2$$

Hence, the new proposed model yields higher expected payoffs for the auctioneer than to typical first price auction without reservation price (model $\mathcal{M}_1$) if and only if:

$$R_3(n)>R_1(n)$$

$$R_{11}(n)+\Delta(n)+R_{32}(n)>2R_{11}(n)$$

$$\Delta(n)>R_{11}(n)-R_{32}(n)$$

$$\Leftrightarrow\frac{1}{2}\frac{3n-1}{n^2(n+1)}>\frac{n-1}{n+1}-\left(\frac{n-1}{n}\right)^2\Leftrightarrow n>-1$$

Therefore, the Competitive advantage mechanism is always superior to the first price model without reservation price.

Similarly, the newly proposed model yields higher expected payoffs for the auctioneer compared to a typical first-price auction with optimal reservation price (model $\mathcal{M}_2$), if and only if:

$$R_3(n)>R_2(n)$$

$$R_{11}(n)+\Delta(n)+R_{32}(n)>2R_{21}(n)$$

$$\Delta(n)>2R_{21}(n)-R_{11}(n)-R_{32}(n)$$

$$\Leftrightarrow\frac{1}{2}\frac{3n-1}{n^2(n+1)}>2\left(\frac{n-1}{n+1}+\frac{1}{2^n}\frac{1}{n+1}\right)-\frac{n-1}{n+1}-\left(\frac{n-1}{n}\right)^2\Leftrightarrow n>3$$

Therefore, the Competitive advantage mechanism is always superior to the first price model with optimal reservation price, as long as $n\geq 4$.