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A THEORY OF AUCTIONS AND COMPETITIVE BIDDING[1]

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A model of competitive bidding is developed in which the winning bidder's payoff may depend upon his personal preferences, the preferences of others, and the intrinsic qualities of the object being sold. In this model, the English (ascending) auction generates higher average prices than does the second-price auction. Also, when bidders are risk-neutral, the second-price auction generates higher average prices than the Dutch and first-price auctions. In all of these auctions, the seller can raise the expected price by adopting a policy of providing expert appraisals of the quality of the objects he sells.

1. INTRODUCTION

THE DESIGN AND CONDUCT of auctioning institutions has occupied the attention of many people over thousands of years. One of the earliest reports of an auction was given by the Greek historian Herodotus, who described the sale of women to be wives in Babylonia around the fifth century B.C. During the closing years of the Roman Empire, the auction of plundered booty was common. In China, the personal belongings of deceased Buddhist monks were sold at auction as early as the seventh century A.D.[2]

In the United States in the 1980's, auctions account for an enormous volume of economic activity. Every week, the U.S. Treasury sells billions of dollars of bills and notes using a sealed-bid auction. The Department of the Interior sells mineral rights on federally-owned properties at auction.[3] Throughout the public and private sectors, purchasing agents solicit delivery-price offers of products ranging from office supplies to specialized mining equipment; sellers auction antiques and artwork, flowers and livestock, publishing rights and timber rights, stamps and wine.

The large volume of transactions arranged using auctions leads one to wonder what accounts for the popularity of such common auction forms as the English auction,[4] the Dutch auction,[5] the first-price sealed-bid auction,[6] and the second-

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[2]These and other historical references can be found in Cassady [2].

[3]On September 30, 1980, U.S. oil companies paid $2.8 billion for drilling rights on 147 tracts in the Gulf of Mexico. The three most expensive individual tracts brought prices of $165 million, $162 million, and $121 million respectively.

[4]The English (ascending, progressive, open, oral) auction is an auction with many variants, some of which are described in Section 5. In the variant we study, the auctioneer calls successively higher prices until only one willing bidder remains, and the number of active bidders is publicly known at all times.

[5]The Dutch (descending) auction, which has been used to sell flowers for export in Holland, is conducted by an auctioneer who initially calls for a very high price and then continuously lowers the price until some bidder stops the auction and claims the flowers for that price.

price sealed-bid auction.[7] What determines which form will (or should) be used in any particular circumstance?

Equally important, but less thoroughly explored, are questions about the relationship between auction theory and traditional competitive theory. One may ask: Do the prices which arise from the common auction forms resemble competitive prices? Do they approach competitive prices when there are many buyers and sellers? In the case of sales of such things as securities, mineral rights, and timber rights, where the bidders may differ in their knowledge about the intrinsic qualities of the object being sold, do prices aggregate the diverse bits of information available to the many bidders (as they do in some rational expectations market equilibrium models)?

In Section 2, we review some important results of the received auction theory, introduce a new general auction model, and summarize the results of our analysis. Section 3 contains a formal statement of our model, and develops the properties of "affiliated" random variables. The various theorems are presented in Sections 4–8. In Section 9, we offer our views on the current state of auction theory. Following Section 9 is a technical appendix dealing with affiliated random variables.

2. AN OVERVIEW OF THE RECEIVED THEORY AND NEW RESULTS[8]

2.1. *The Independent Private Values Model*

Much of the existing literature on auction theory analyzes the *independent private values model.* In that model, a single indivisible object is to be sold to one of several bidders. Each bidder is risk-neutral and knows the value of the object to himself, but does not know the value of the object to the other bidders (this is the *private values* assumption). The values are modeled as being independently drawn from some continuous distribution. Bidders are assumed to behave competitively;[9] therefore, the auction is treated as a noncooperative game among the bidders.[10]

At least seven important conclusions emerge from the model. The first of these is that the Dutch auction and the first-price auction are strategically equivalent.

[6]The first-price auction is a sealed-bid auction in which the buyer making the highest bid claims the object and pays the amount he has bid.

[7]The second-price auction is a sealed-bid auction in which the buyer making the highest bid claims the object, but pays only the amount of the second highest bid. This arrangement does not necessarily entail any loss of revenue for the seller, because the buyers in this auction will generally place higher bids than they would in the first-price auction.

[8]A more thorough survey of the literature is given by Engelbrecht-Wiggans [**4**]. A comprehensive bibliography of bidding, including almost 500 titles, has been compiled by Stark and Rothkopf [**26**].

[9]Situations in which bidders collude have received no attention in theoretical studies, despite many allegations of collusion, particularly in bidding for timber rights (Mead [**14**]).

[10]The case in which several identical objects are offered for sale with a limit of one item per bidder has also been analyzed (Ortega-Reichert [**22**], Vickrey [**30**]). All of the results discussed below have natural analogues in that more general setting.

Another variation, in which the bidders' private valuations are drawn from a common but unknown distribution, has been treated by Wilson [**34**].

Recall that in a Dutch auction, the auctioneer begins by naming a very high price and then lowers it continuously until some bidder stops the auction and claims the object for that price. An insight due to Vickrey [**29**] is that the decision faced by a bidder with a particular valuation is essentially static, i.e. the bidder must choose the price level at which he will claim the object if it has not yet been claimed. The winning bidder will be the one who chooses the highest level, and the price he pays will be equal to that amount. This, of course, is also the way the winner and price are determined in the sealed-bid first-price auction. Thus, the sets of strategies and the mapping from strategies to outcomes are the same for both auction forms. Consequently, the equilibria of the two auction games must coincide.

The second conclusion is that—in the context of the private values model—the second-price sealed-bid auction and the English auction are equivalent, although in a weaker sense than the "strategic equivalence" of the Dutch and first-price auctions. Recall that in an English auction, the auctioneer begins by soliciting bids at a low price level, and he then gradually raises the price until only one willing bidder remains. In this setting, a bidder's strategy must specify, for each of his possible valuations, whether he will be active at any given price level, as a function of the previous activity he has observed during the course of the auction. However, if a bidder knows the value of the object to himself, he has a straightforward dominant strategy, which is to bid actively until the price reaches the value of the object to him. Regardless of the strategies adopted by the other bidders, this simple strategy will be an optimal reply.

Similarly, in the second-price auction, if a bidder knows the value of the object to himself, then his dominant strategy is to submit a sealed bid equal to that value. Thus, in both the English and second-price auctions, there is a unique dominant-strategy equilibrium. In both auctions, at equilibrium, the winner will be the bidder who values the object most highly, and the price he pays will be the value of the object to the bidder who values it second-most highly. In that sense, the two auctions are equivalent. Note that this argument requires that each bidder know the value of the object to himself.[11] If what is being sold is the right to extract minerals from a property, where the amount of recoverable minerals is unknown, or if it is a work of art, which will be enjoyed by the buyer and then eventually resold for some currently undetermined price, then this equivalence result generally does not apply.

A third result is that the outcome (at the dominant-strategy equilibrium) of the English and second-price auctions is Pareto optimal; that is, the winner is the bidder who values the object most highly. This conclusion follows immediately from the argument of the preceding paragraph and, like the first two results, does not depend on the symmetry of the model. In symmetric models the Dutch and first-price auctions also lead to Pareto optimal allocations.

[11] In contrast, the argument concerning the strategic equivalence of the Dutch and first-price auctions does not require any assumptions about the values to the bidders of various outcomes. In particular, it does not require that a bidder know the value of the object to himself.

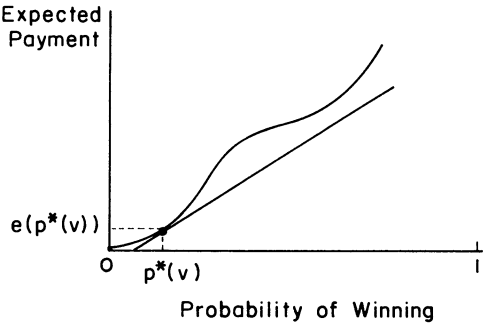

FIGURE 1.

A fourth result is that in the independent private values model, all four auction forms lead to identical expected revenues for the seller (Ortega-Reichert [**22**], Vickrey [**30**]). This result remained a puzzle until recently, when an application of the self-selection approach cast it in a new light (Harris and Raviv [**8**], Myerson [**21**], Riley and Samuelson [**24**]). That approach views a bidder's decision problem (when the strategies of the other bidders are fixed) as one of choosing, through his action, a probability p of winning and a corresponding expected payment $e(p)$. (We take $e(p)$ to be the lowest expected payment associated with an action which obtains the object with probability p.) It is important to notice that, because of the independence assumption, the set of $(p, e(p))$ pairs that are available to the bidder depends only on the rules of the auction and the strategies of the others, and not on his private valuation of the object.

Figure 1 displays a typical bidding decision faced by a bidder who values the prize at v. The curve consists of the set of $(p, e(p))$ pairs among which he must choose.[12] Since the bidder's expected utility from a point (p, e) is $v \cdot p - e$, his indifference curves are straight lines with slope v. Let $p^*(v)$ denote the optimal choice of p for a bidder with valuation v. It is clear from the figure that p^* must be nondecreasing.

In Figure 1, the tangency condition is $e'(p^*(v)) = v$. Similarly, when the indifference line has multiple points of tangency, a small increase in v causes a jump Δp^* in p^* and a corresponding jump $\Delta e = v \cdot \Delta p^*$ in $e(p^*(v))$. Hence we can conclude quite generally that $e(p^*(v)) = e(p^*(0)) + \int_0^v t\, dp^*(t)$. It then follows that the seller's expected revenue from a bidder depends on the rules of the auction only to the extent that the rules affect either $e(p^*(0))$ or the p^* function. Notice, in particular, that all auctions which always deliver the prize to the highest evaluator have the same p^* function for all bidders. That observation, together with the fact that at the dominant-strategy equilibrium the second-price

[12] In general, the $(p, e(p))$-curve need not be continuous; there may even be values of p for which *no* $(p, e(p))$ pair is available. However, there will *always* be a point $(0, e(0))$ on the curve, with $e(0) \leq 0$, for the bidder is free to abstain from participation. The quantity $e(0)$ will be negative only if the seller at times provides subsidies to losing bidders.

auction yields a price equal to the second-highest valuation, leads to the fifth result.

THEOREM 0: *Assume that a particular auction mechanism is given, that the independent private values model applies, and that the bidders adopt strategies which consititute a noncooperative equilibrium. Suppose that at equilibrium the bidder who values the object most highly is certain to receive it, and that any bidder who values the object at its lowest possible level has an expected payment of zero. Then the expected revenue generated for the seller by the mechanism is precisely the expected value of the object to the second-highest evaluator.*

At the symmetric equilibria of the English, Dutch, first-price, and second-price auctions, the conditions of the theorem are satisfied. Consequently, the expected selling price is the same for all four mechanisms; this is the so-called "revenue-equivalence" result. It should be noted that Theorem 0 has an attractive economic interpretation. No matter what competitive mechanism is used to establish the selling price of the object, on average the sale will be at the lowest price at which supply (a single unit) equals demand.

The self-selection approach has also been applied to the problem of designing auctions to maximize the seller's expected revenue (Harris and Raviv [**8**], Myerson [**21**], Riley and Samuelson [**24**]). The problem is formulated very generally as a constrained optimal control problem, where the control variables are the pairs $(p_i^*(\cdot), e_i(p_i^*(0)))$. As might be expected, the form of the optimal auction depends on the underlying distribution of bidder valuations. One remarkable conclusion emerging from the analysis is this: For many common sample distributions—including the normal, exponential, and uniform distributions—the four standard auction forms with suitably chosen reserve prices or entry fees are optimal auctions.

The seventh and last result in this list arises in a variation of the model where either the seller or the buyers are risk averse. In that case, the seller will strictly prefer the Dutch or first-price auction to the English or second-price auction (Harris and Raviv [**8**], Holt [**9**], Maskin and Riley [**11**], Matthews [**13**]).

2.2. *Oil, Gas, and Mineral Rights*

The private values assumption is most nearly satisfied in auctions for nondurable consumer goods. The satisfaction derived from consuming such goods is reasonably regarded as a personal matter, so it is plausible that a bidder may know the value of the good to himself, and may allow that others could value the good differently.

In contrast, consider the situation in an auction for mineral rights on a tract of land where the value of the rights depends on the unknown amount of recoverable ore, its quality, its ease of recovery, and the prices that will prevail for the processed mineral. To a first approximation, the values of these mineral rights to

the various bidders can be regarded as equal, but bidders may have differing estimates of the common value.

Suppose the bidders make (conditionally) independent estimates of this common value V. Other things being equal, the bidder with the largest estimate will make the highest bid. Consequently, even if all bidders make unbiased estimates, the winner will find that he had overestimated (on average) the value of the rights he has won at auction. Petroleum engineers (Capen, Clapp, and Campbell [1]) have claimed that this phenomenon, known as the *winner's curse*, is responsible for the low profits earned by oil companies on offshore tracts in the 1960's.

The model described above, in which risk-neutral bidders make independent estimates of the common value where the estimates are drawn from a single underlying distribution parameterized by V, can be called the *mineral rights model* or the *common value model*. The equilibrium of the first-price auction for this model has been extensively studied (Maskin and Riley [11], Milgrom [15, 16], Milgrom and Weber [20], Ortega-Reichert [22], Reece [23], Rothkopf [25], Wilson [34]). Among the most interesting results for the mineral rights model are those dealing with the relations between information, prices, and bidder profits.

For example, consider the information that is reflected in the price resulting from a mineral rights auction. It is tempting to think that this price cannot convey more information than was available to the winning bidder, since the price is just the amount that he bid. This reasoning, however, is incorrect. Since the winning bidder's estimate is the maximum among all the estimates, the winning bid conveys a bound on all the loser's estimates. When there are many bidders, the price conveys a bound on many estimates, and so can be very informative. Indeed, let $f(x \mid v)$ be the density of the distribution of a bidder's estimate when $V = v$. A property of many one-parameter sampling distributions is that for $v_1 < v_2$, $f(x \mid v_1)/f(x \mid v_2)$ declines as x increases.[13] If this ratio approaches zero, then the equilibrium price in a first-price auction with many bidders is a consistent estimator of the value V, even if no bidder can estimate V closely from his information alone (Milgrom [15, 16], Wilson [34]). Thus, the price can be surprisingly effective in aggregating private information.

Several results and examples suggest that a bidder's expected profits in a mineral rights auction depend more on the privacy of his information than on its accuracy as information about V. For example, in the first-price auction a bidder whose information is also available to some other bidder must have zero expected profits at equilibrium (Engelbrecht-Wiggans, Milgrom, and Weber [5], Milgrom [15]). Thus, if two bidders have access to the same estimate of V and a third bidder has access only to some less informative but independent estimate, then the two relatively well-informed bidders must have zero expected profits, but the more poorly-informed bidder may have positive expected profits. Related results appear in Milgrom [15 and 17] and as Theorem 7 of this paper.

[13] This property is known to statisticians as the *monotone likelihood ratio property* (Tong [27]). Its usefulness for economic modelling has been elaborated by Milgrom [18].

2.3. *A General Model*

Consider the issues that arise in attempting to select an auction to use in selling a painting. If the independent private values model is to be applied, one must make two assumptions: that each bidder knows his value for the painting, and that the values are statistically independent. The first assumption rules out the possibilities: (i) that the painting may be resold later for an unknown price, (ii) that there may be some "prestige" value in owning a painting which is admired by other bidders, and (iii) that the authenticity of the painting may be in doubt. The second assumption rules out the possibility that several bidders may have relevant information concerning the painting's authenticity, or that a buyer, thinking that the painting is particularly fine, may conclude that other bidders also are likely to value it highly. Only if these assumptions are palatable can the theory be used to guide the seller's choice of an auction procedure. Even in this case, however, little guidance is forthcoming: the theory predicts that the four most common auction forms lead to the same expected price.

Unlike the private values theory, the common value theory allows for statistical dependence among bidders' value estimates, but offers no role for differences in individual tastes. Furthermore, the received theory offers no basis for choosing among the first-price, second-price, Dutch, and English auction procedures.

In this paper, we develop a general auction model for risk-neutral bidders which includes as special cases the independent private values model and the common value model, as well as a range of intermediate models which can better represent, for example, the auction of a painting. Despite its generality, the model yields several testable predictions. First, the Dutch and first-price auctions are strategically equivalent in the general model, just as they were in the private values model. Second, when bidders are uncertain about their value estimates, the English and second-price auctions are not equivalent: the English auction generally leads to larger expected prices. One explanation of this inequality is that when bidders are uncertain about their valuations, they can acquire useful information by scrutinizing the bidding behavior of their competitors during the course of an English auction. That extra information weakens the winner's curse and leads to more aggressive bidding in the English auction, which accounts for the higher expected price.

A third prediction of the model is that when the bidders' value estimates are statistically dependent, the second-price auction generates a higher average price than does the first-price auction. Thus, the common auction forms can be ranked by the expected prices they generate. The English auction generates the highest prices followed by the second-price auction and, finally, the Dutch and first-price auctions. This may explain the observation that "an estimated 75 per cent, or even more, of all auctions in the world are conducted on an ascending-bid basis" (Cassady [**2**, page 66]).

Suppose that the seller has access to a private source of information. Further, suppose that he can commit himself to any policy of reporting information that he chooses. Among the possible policies are: (i) concealment (never report any

information), (ii) honesty (always report all information completely), (iii) censoring (report only the most favorable information), (iv) summarizing (report only a rough summary statistic), and (v) randomizing (add noise to the data before reporting).

The fourth conclusion of our analysis is that for the first-price, second-price, and English auctions policy, (ii) maximizes the expected price: Honesty is the best policy.

The general model and its assumptions are presented in Section 3. The analysis of the model is driven by the assumption that the bidders' valuations are *affiliated*. Roughly, this means that a high value of one bidder's estimate makes high values of the others' estimates more likely. This assumption, though restrictive, accords well with the qualitative features of the situations we have described.

Sections 4 through 6 develop our principal results concering the second-price, English, and first-price auction procedures.

In Section 7, we modify the general model by introducing reserve prices and entry fees. The introduction of a positive reserve price causes the number of bidders actually submitting bids to be random, but this does not significantly change the analysis of equilibrium strategies nor does it alter the ranking of the three auction forms as revenue generators. However, it does change the analysis of information reporting by the seller, because the number of competitors who are willing to bid at least the reserve price will generally depend on the details of the report: favorable information will attract additional bidders and unfavorable information will discourage them. The seller can offset that effect by adjusting the reserve price (in a manner depending on the particular realization of his information variable) so as to always attract the same set of bidders. When this is done, the information-release results mentioned above continue to hold.

When both a reserve price and an entry fee are used, a bidder will participate in the auction if and only if his expected profit from bidding (given the reserve price) exceeds the entry fee. In particular, he will participate only if his value estimate exceeds some minimum level called the *screening level.* The most tractable case for analysis arises when the "only if" can be replaced by "if and only if," that is, when every bidder whose value estimate exceeds the screening level participates: we call that case the *regular* case. The case of a zero entry fee is always regular.

For each type of auction we study, any particular screening level x^* can be achieved by a continuum of different combinations (r, e) of reserve prices and entry fees. We show that if (r, e) and $(\bar{r}, \bar{e})$ are two such combinations with $e > \bar{e}$, and if the auction corresponding to (r, e) is regular, then the auction corresponding to $(\bar{r}, \bar{e})$ is also regular but generates lower expected revenues than the (r, e)-auction. Therefore, so long as regularity is preserved and the screening level is held fixed, it pays to raise entry fees and reduce reserve prices.

In Section 8, we consider another variation of the general model, in which bidders are risk-averse. Recall that in the independent private values model with

risk aversion, the first-price auction yields a larger expected price than do the second-price and English auctions. In our more general model, no clear qualitative comparison can be made between the first-price and second-price auctions in the presence of risk aversion, and all that can be generally said about reserve prices and entry fees in the first-price auction is that the revenue-maximizing fee is positive (cf. Maskin and Riley [**11**]). With constant absolute risk aversion, however, both the results that the English auction generates higher average prices than the second-price auction, and that the best information-reporting policy for the seller in either of these two auctions is to reveal fully his information, retain their validity.

3. THE GENERAL SYMMETRIC MODEL

Consider an auction in which n bidders compete for the possession of a single object. Each bidder possesses some information concerning the object up for sale; let $X = (X_1, \ldots, X_n)$ be a vector, the components of which are the real-valued *informational variables*[14] (or *value estimates*, or *signals*) observed by the individual bidders. Let $S = (S_1, \ldots, S_m)$ be a vector of additional real-valued variables which influence the value of the object to the bidders. Some of the components of S might be observed by the seller. For example, in the sale of a work of art, some of the components may represent appraisals obtained by the seller, while other components may correspond to the tastes of art connoisseurs not participating in the auction; these tastes could affect the resale value of the object.

The actual value of the object to bidder i—which may, of course, depend on variables not observed by him at the time of the auction—will be denoted by $V_i = u_i(S, X)$. We make the following assumptions:

ASSUMPTION 1: There is a function u on $\mathbb{R}^{m+n}$ such that for all i, $u_i(S, X) = u(S, X_i, \{X_j\}_{j \neq i})$. Consequently, all of the bidders' valuations depend on S in the same manner, and each bidder's valuation is a symmetric function of the other bidders' signals.

ASSUMPTION 2: The function u is nonnegative, and is continuous and nondecreasing in its variables.

ASSUMPTION 3: For each i, $E[V_i] < \infty$.

[14]To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders. The derivation of such a statistic from several separate pieces of information is in general a difficult task (see, for example, the discussion in Engelbrecht-Wiggans and Weber [7]). It is in the light of these difficulties that we choose to view each X_i as a "value estimate," which may be correlated with the "estimates" of others but is the only piece of information available to bidder i.

Both the private values model and the common value model involve valuations of this form. In the first case, $m = 0$ and each $V_i = X_i$; in the second case, $m = 1$ and each $V_i = S_1$.

Throughout the next four sections, we assume that the bidders' valuations are in monetary units, and that the bidders are neutral in their attitudes towards risk. Hence, if bidder i receives the object being sold and pays the amount b, his payoff is simply $V_i - b$.

Let $f(s, x)$ denote the joint probability density[15] of the random elements of the model. We make two assumptions about the joint distribution of S and X:

ASSUMPTION 4: f is symmetric in its last n arguments.

ASSUMPTION 5: The variables $S_1, \ldots, S_m, X_1, \ldots, X_n$ are affiliated.

A general definition of affiliation is given in the Appendix. For variables with densities, the following simple definition will suffice.

Let z and z' be points in $\mathbb{R}^{m+n}$. Let $z \vee z'$ denote the component-wise maximum of z and z', and let $z \wedge z'$ denote the component-wise minimum. We say that the variables of the model are *affiliated* if, for all z and z',

$$f(z \vee z')f(z \wedge z') \geqq f(z)f(z'). \tag{2}$$

Roughly, this condition means that large values for some of the variables make the other variables more likely to be large than small.

We call inequality (2) the "affiliation inequality" (though it is also known as the "FKG inequality" and the "MTP_2 property"), and a function f satisfying (2) is said to be "affiliated." Some consequences of affiliation are discussed by Karlin and Rinott [**10**] and by Tong [**27**], and related results are reported by Milgrom [**18**] and Whitt [**32**]. For our purposes, the major results are those given by Theorems 1–5 below.

THEOREM 1: *Let $f: \mathbb{R}^k \to \mathbb{R}$.* (i) *If f is strictly positive and twice continuously differentiable, then f is affiliated if and only if for $i \neq j$, $\partial^2 \ln f / \partial z_i \partial z_j \geq 0$.* (ii) *If $f(z) = g(z)h(z)$ where g and h are nonnegative and affiliated, then f is affiliated.*

A proof of part (i) can be found in Topkis [**28**, p. 310]. Part (ii) is easily checked.

[15]This assumption—that the joint distribution of the various signals has an associated density—substantially simplifies the development of our results by making the statement of later assumptions simpler, and by ensuring the existence of equilibrium points in pure strategies. All of the results in this paper, except for the explicit characterizations of equilibrium strategies, continue to hold when this assumption is eliminated. In the general case, equilibrium strategies may involve randomization. These randomized strategies can be obtained directly, or indirectly as the limits of sequences of pure equilibrium strategies of the games studied here, using techniques developed in Engelbrecht-Wiggans, Milgrom, and Weber [**5**], Milgrom [**17**], and Milgrom and Weber [**19**].

In the independent private values model, the only random variables are $X_1, \ldots, X_n$, and they are statistically independent. For this case, (2) always holds with equality: Independent variables are always affiliated.

In the mineral rights model, let $g(x_i \mid s)$ denote the conditional density of any X_i given the common value S and let h be the marginal density of S. Then $f(s,x) = h(s)g(x_i \mid s) \ldots g(x_n \mid s)$. Assume that the density g has the monotone likelihood ratio property; that is, assume that $g(x \mid s)$ satisfies (2).[16] It then follows from Theorem 1 (ii) that f satisfies (2). Consequently, for the case of densities g with the monotone likelihood ratio property, the mineral rights model fits our formulation.

The affiliation assumption also accommodates other forms of the density f. For example, it accommodates a number of variations of the mineral rights model in which the bidders' estimation errors are positively correlated. And, if the inequality in (2) is strict, it formalizes the assumption that in an auction for a painting, a bidder who finds the painting very beautiful will expect others to admire it, too.

In this symmetric bidding environment, we identify competitive behavior with symmetric Nash equilibrium behavior. We will find that, at equilibrium, bidders with higher estimates tend to make higher bids. Consequently, we shall need to understand the properties of the distribution of the highest estimates.

Let $Y_1, \ldots, Y_{n-1}$ denote the largest, ..., smallest estimates from among $X_2, \ldots, X_n$. Then, using (1) and the symmetry assumption, we can rewrite bidder 1's value as follows:

$$V_1 = u(S_1, \ldots, S_m, X_1, Y_1, \ldots, Y_{n-1}). \tag{3}$$

The joint density of $S_1, \ldots, S_m, X_1, Y_1, \ldots, Y_{n-1}$ is

$$(n-1)!\, f(x_1, \ldots, s_m, x_1, y_1, \ldots, y_{n-1}) 1_{\{y_1 \geqq y_2 \geqq \cdots \geqq y_{n-1}\}}, \tag{4}$$

where the last term is an indicator function. Applying Theorem 1 (ii) to (4), we have the following result.

THEOREM 2: *If f is affiliated and symmetric in $X_2, \ldots, X_n$, then $S_1, \ldots, S_m$, $X_1, Y_1, \ldots, Y_{n-1}$ are affiliated.*

The following additional results, which are used repeatedly, are derived in the Appendix.

THEOREM 3: *If $Z_1, \ldots, Z_k$ are affiliated and $g_1, \ldots, g_k$ are all nondecreasing functions (or all nonincreasing functions), then $g_1(Z_1), \ldots, g_k(Z_k)$ are affiliated.*

[16]The density g has the monotone likelihood ratio property if for all $s' > s$ and $x' > x$, $g(x \mid s)/g(x \mid s') \geq g(x' \mid s)/g(x' \mid s')$. This is equivalent to the affiliation inequality: $g(x \mid s)g(x' \mid s') \geq g(x' \mid s)g(x \mid s')$.

THEOREM 4: *If $Z_1, \ldots, Z_k$ are affiliated, then $Z_1, \ldots, Z_{k-1}$ are affiliated.*

THEOREM 5: *Let $Z_1, \ldots, Z_k$ be affiliated and let H be any nondecreasing function. Then the function h defined by*

$$h(a_1, b_1; \ldots; a_k, b_k)$$
$$= E\big[H(Z_1, \ldots, Z_k) \mid a_1 \le Z_1 \le b_1, \ldots, a_k \le Z_k \le b_k\big]$$

is nondecreasing in all of its arguments. In particular, the functions

$$h_l(z_1, \ldots, z_l) = E\big[H(Z_1, \ldots, Z_k) \mid z_1, \ldots, z_l\big]$$

for $l = 1, \ldots, k$ are all nondecreasing.

In view of Theorems 2 and 5, we can conclude that the function $E[V_1 \mid X_1 = x, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}]$ is nondecreasing in x. To simplify later proofs, we add the nondegeneracy assumption that this function is strictly increasing in x. All of our results can be shown to hold without this extra assumption.

4. SECOND-PRICE AUCTIONS[17]

In the second-price auction game, a strategy for bidder i is a function mapping his value estimate x_i into a bid $b = b_i(x_i) \geqq 0$. Since the auction is symmetric, let us focus our attention on the bidding decision faced by bidder 1.

Suppose that the bidders $j \neq 1$ adopt strategies b_j. Then the highest bid among them will be $W = \max_{j \neq 1} b_j(X_j)$ which, for fixed strategies b_j, is a random variable. Bidder 1 will win the second-price auction if his bid b exceeds W, and W is the price he will pay if he wins. Thus, his decision problem is to choose a bid b to solve

$$\max_b E\big[(V_1 - W)1_{\{W<b\}} \mid x_1\big].$$

If $b_1(x_1)$ solves this problem for every value of x_1, then the strategy b_1 is called a *best reply* to $b_2, \ldots, b_n$. If each b_i in an n-tuple $(b_1, \ldots, b_n)$ is a best reply to the remaining $n - 1$ strategies, then the n-tuple is called an *equilibrium point*.

Let us define a function $v : \mathbb{R}^2 \to \mathbb{R}$ by $v(x, y) = E[V_1 \mid X_1 = x, Y_1 = y]$. In view of (3) and Theorems 2 and 5, v is nondecreasing. Our nondegeneracy assumption ensures that v is strictly increasing in its first argument.

THEOREM 6: *Let $b^*(x) = v(x, x)$. Then the n-tuple of strategies $(b^*, \ldots, b^*)$ is an equilibrium point of the second-price auction.*

[17] Our basic analysis of the second-price auction is very similar to that given in Milgrom [17], although the present set-up is a bit different. Theorems 6 and 7 were first proved in that reference.

PROOF: Since b^* is increasing, $W = b^*(Y_1)$. So bidder 1's conditional expected payoff when he bids b is

$$\begin{aligned}
&E\big[(V_1 - b^*(Y_1))1_{\{b^*(Y_1)<b\}} \,|\, X_1 = x\big] \\
&\quad = E\big[E\big[(V_1 - v(Y_1, Y_1))1_{\{b^*(Y_1)<b\}} \,|\, X_1, Y_1\big] \,|\, X_1 = x\big] \\
&\quad = E\big[(v(X_1, Y_1) - v(Y_1, Y_1))1_{\{b^*(Y_1)<b\}} \,|\, X_1 = x\big] \\
&\quad = \int_{-\infty}^{b^{*-1}(b)} \big[v(x, \alpha) - v(\alpha, \alpha)\big] f_{Y_1}(\alpha \,|\, x)\, d\alpha,
\end{aligned}$$

where $f_{Y_1}(\cdot \,|\, x)$ is the conditional density of Y_1 given $X_1 = x$. Since v is increasing in its first argument, the integrand is positive for $\alpha < x$ and negative for $\alpha > x$. Hence, the integral is maximized by choosing b so that $b^{*-1}(b) = x$, i.e., $b = b^*(x)$. This proves that b^* is a best reply for bidder 1. *Q.E.D.*

An important special case arises if we assume that $V_1 = V_2 = \cdots = V_n = V$. We call this the *generalized mineral rights model.* (It differs from the mineral rights model in not requiring the bidders' estimates of V to be conditionally independent.) Suppose that, in this context, we introduce an $(n+1)$st bidder with an estimate X_{n+1} of the common value V. We say that X_{n+1} is a *garbling* of (X_1, Y_1) if the joint density of $(V, X_1, \ldots, X_n, X_{n+1})$ can be written as $g(V, X_1, \ldots, X_n) \cdot h(X_{n+1} \,|\, X_1, Y_1)$. For example, if bidder $n+1$ bases his estimate X_{n+1} only on information that was also available to bidder 1, this condition would hold.

THEOREM 7: *For the generalized mineral rights model, if X_{n+1} is a garbling of (X_1, Y_1), then bidder $n+1$ has no strategy that earns a positive expected payoff when bidders $1, \ldots, n$ use $(b^*, \ldots, b^*)$. Consequently, in this $(n+1)$-bidder second-price auction, the $(n+1)$-tuple $(b^*, \ldots, b^*, b_{n+1})$ where $b_{n+1} \equiv 0$ is an equilibrium point.*

PROOF: Let $Z = \max(X_1, Y_1)$. If bidder $n+1$ observes X_{n+1} and then makes a winning bid b, then his conditional expected payoff is

$$\begin{aligned}
&E\big[(V - b^*(Z)) \,|\, X_{n+1}, \{b^*(Z) < b\}\big] \\
&\quad = E\big[E\big[V - b^*(Z) \,|\, X_1, Y_1, X_{n+1}\big] \,|\, X_{n+1}, \{b^*(Z) < b\}\big] \\
&\quad = E\big[v(X_1, Y_1) - v(Z, Z) \,|\, X_{n+1}, \{b^*(Z) < b\}\big].
\end{aligned}$$

The last equality uses the fact that $E[V \,|\, X_1, Y_1, X_{n+1}] = E[V \,|\, X_1, Y_1]$, a consequence of the garbling assumption. Since v is nondecreasing, $v(X_1, Y_1) - v(Z, Z) \leqq 0$, so the last expectation is nonpositive. *Q.E.D.*

Now consider how the equilibrium is affected when the seller publicly reveals some information X_0 (which is affiliated with all the other random elements of the model). We shall assume the seller's revelations are credible.[18]

Define a function $w: \mathbb{R}^3 \to \mathbb{R}$ by $w(x, y; z) = E[V_1 \mid X_1 = x, Y_1 = y, X_0 = z]$. By Theorems 2 and 5, w is nondecreasing. After X_0 is publicly announced, a new conditional joint density $f(s_1, \ldots, s_m, x_1, \ldots, x_n \mid x_0)$ applies to the random elements of the model, and it is straightforward to verify that the conditional density satisfies the affiliation inequality. So, carrying out the same analysis as before, there is an equilibrium $(\hat{b}, \ldots, \hat{b})$ given by $\hat{b}(x; x_0) = w(x, x; x_0)$. Note that this time a strategy maps two variables, representing private and public information, into a bid. For any fixed value of X_0, the equilibrium strategy is a function of a single variable and is similar in form to b^*.

Let R_N be the expected selling price when no public information is revealed and let R_I be the expected price when X_0 is made public.

THEOREM 8: *The expected selling prices are as follows*:

$$R_N = E\big[v(Y_1, Y_1) \mid \{X_1 > Y_1\}\big],$$

$$R_I = E\big[w(Y_1, Y_1; X_0) \mid \{X_1 > Y_1\}\big].$$

Revealing information publicly raises revenues, that is, $R_I \geqq R_N$.

PROOF: Recall that $v(Y_1, Y_1)$ is the price paid when bidder 1 wins. Thus, R_N is the expected price paid by bidder 1 when he wins. By symmetry, it is the expected price, regardless of the winner's identity. The same argument applies to R_I.

Next, note the following identities.

$$\begin{aligned} v(x, y) &= E\big[V_1 \mid X_1 = x, Y_1 = y\big] \\ &= E\big[E\big[V_1 \mid X_1, Y_1, X_0\big] \mid X_1 = x, Y_1 = y\big] \\ &= E\big[w(X_1, Y_1; X_0) \mid X_1 = x, Y_1 = y\big]. \end{aligned}$$

For $x > y$, we apply Theorems 2, 4, and 5 to get:

$$\begin{aligned} v(y, y) &= E\big[w(X_1, Y_1; X_0) \mid X_1 = y, Y_1 = y\big] \\ &= E\big[w(Y_1, Y_1; X_0) \mid X_1 = y, Y_1 = y\big] \\ &\leqq E\big[w(Y_1, Y_1; X_0) \mid X_1 = x, Y_1 = y\big]. \end{aligned}$$

[18] This might be the case if, for example, there were some effective recourse available to the buyer if the seller made a false announcement, or if the seller were an institution, like an auction house, which valued its reputation for truthfulness.

So,

$$R_N = E\big[v(Y_1, Y_1) \,|\, \{X_1 > Y_1\}\big]$$
$$\leqq E\big[E\big[w(Y_1, Y_1; X_0) \,|\, X_1, Y_1\big] \,|\, \{X_1 > Y_1\}\big]$$
$$= E\big[w(Y_1, Y_1; X_0) \,|\, \{X_1 > Y_1\}\big] = R_I. \qquad \textit{Q.E.D.}$$

Theorem 8 indicates that publicly revealing the information X_0 is better, on average, than revealing no information. One might wonder whether it would be better still to censor information sometimes, i.e., to report X_0 only when it exceeds some critical level. Of course, if this policy of the seller were known, rational bidders would correctly interpret the absence of any report as a bad sign.

There are many possible information revelation policies. If one assumes that the bidders know the information policy, then one can also assume without loss of generality that the seller always makes some report, though that report may consist of a blank page. Let Z be a random variable, uniformly distributed on $[0, 1]$ and independent of the other variables of the model. We formulate the seller's report very generally as $X_0' = r(X_0, Z)$, i.e., the seller's report may depend both on his information and the spin of a roulette wheel. We call r the seller's *reporting policy*.

THEOREM 9: *In the second-price auction, no reporting policy leads to a higher expected price than the policy of always reporting* X_0.

PROOF: Let r be any reporting policy and let $X_0' = r(X_0, Z)$. The conditional distribution of X_0', given the original variables (S, X), depends only on X_0. We denote the conditional density (if one exists) by $g(X_0' \,|\, X_0)$ and the marginal density by $g(X_0')$. For any realization x_0' of X_0', the corresponding conditional joint density[19] of (S, X) is $f(s, x)g(x_0' \,|\, x_0)/g(x_0')$, which satisfies the affiliation inequality in (s, x) since f does, by Theorem 1. Therefore, by Theorem 8, revealing X_0 further raises expected revenues. But revealing both X_0 and X_0' leads to the same equilibrium bidding as revealing just X_0, so the result follows. *Q.E.D.*

5. ENGLISH AUCTIONS

There are many variants of the English auction. In some, the bids are called by the bidders themselves, and the auction ends when no one is willing to raise the

[19] If $G_{X_0}(\cdot \,|\, X_0')$ denotes the conditional distribution of X_0 given X_0', then the variables $S_1, \ldots, S_m$, $X_0, X_1, \ldots, X_n$ always will have a density with respect to the product measure $M^m \times G(\cdot \,|\, X_0') \times M^n$, where M is Lebesgue measure, and the density always will have the form $f(s, x)g(x_0 \,|\, x_0')/f(x_0)$. A density with respect to *any* product measure suffices for our analysis, so the theorem is proved by our argument.

bid.[20] In others, the auctioneer calls the bids, and a willing bidder indicates his assent by some slight gesture, usually in a way that preserves his anonymity. Cassady [2] has described yet another variant, used in Japan, in which the price is posted using an electronic display. In that variant, the price is raised continuously, and a bidder who wishes to be active at the current price depresses a button. When he releases the button, he has withdrawn from the auction. These three forms of the English auction correspond to three quite different games. The game model developed in this section corresponds most closely to the Japanese variant. We assume that both the price level and the number of active bidders are continuously displayed. We use the term "English auction" to designate this variant.

In the English auction with only two bidders, each bidder's strategy can be completely described by a single number which specifies how high to compete before ceding the contest to the other bidder. The bidder selecting the higher number wins, and he pays a price equal to the other bidder's number. Thus, with only two bidders, the English and second-price auctions are strategically equivalent. When there are three or more bidders, however, the bidding behavior of those who drop out early in an English auction can convey information to those who keep bidding, and our model of the auction as a game must account for that possibility.

We idealize the auction as follows. Initially, all bidders are active at a price of zero. As the auctioneer raises the price, bidders drop out one by one. No bidder who has dropped out can become active again. After any bidder quits, all remaining active bidders know the price at which he quit.

A strategy for bidder i specifies whether, at any price level p, he will remain active or drop out, as a function of his value estimate, the number of bidders who have quit the bidding, and the levels at which they quit. Let k denote the number of bidders who have quit and let $p_1 \leqq \cdots \leqq p_k$ denote the levels at which they quit. Then bidder i's strategy can be described by functions $b_{ik}(x_i \mid p_1, \ldots, p_k)$ which specify the price at which bidder i will quit if, at that point, k other bidders have quit at the prices $p_1, \ldots, p_k$. It is natural to require that $b_{ik}(x_i \mid p_1, \ldots, p_k)$ be at least p_k.

Now consider the strategy $b^* = (b_0^*, \ldots, b_{n-2}^*)$ defined iteratively as follows.

$$b_0^*(x) = E\left[V_1 \mid X_1 = x, Y_1 = x, \ldots, Y_{n-1} = x \right]. \tag{5}$$

$$\begin{aligned} b_k^*(x \mid p_1, \ldots, p_k) = E\big[V_1 \mid X_1 = x,\ & Y_1 = x, \ldots, Y_{n-k-1} = x, \\ & b_{k-1}^*(Y_{n-k} \mid p_1, \ldots, p_{k-1}) = p_k, \ldots, \\ & b_0^*(Y_{n-1}) = p_1 \big]. \end{aligned} \tag{6}$$

[20] A model in which the bidders call the bids has been analyzed by Wilson [33].

The component strategies reflect a kind of myopic bidding behavior. Suppose, for example, that $k = 0$, i.e., no bidder has quit yet. Suppose, too, that the price has reached the level $b_0^*(y)$ and that bidder 1 has observed $X_1 = x$. If bidders $2, \ldots, n$ were to quit instantly, then bidder 1 could infer from this behavior that $Y_1 = \cdots = Y_{n-1} = y$. In that case, he would estimate his payoff to be $E[V_1 | X_1 = x, Y_1 = y, \ldots, Y_{n-1} = y] - b_0^*(y)$. By (5) and Theorem 5, that difference is positive if $x > y$ and negative if $x < y$. Thus, b_0^* calls for bidder 1 to remain active until the price rises to the point where he would be just indifferent between winning and losing at that price. The other strategies b_k^* have similar interpretations, but they assume that bidders infer whatever they can from the quitting prices of those who are no longer active.

THEOREM 10: *The n-tuple* $(b^*, \ldots, b^*)$ *is an equilibrium point of the English auction game.*

PROOF: It is straightforward to verify from (5) and (6) that each b_k^* is increasing in its first argument. Hence, if bidders $2, \ldots, n$ adopt b^* and bidder 1 wins the auction, the price he will pay is $E[V_1 | X_1 = y_1, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}]$ where $y_1, \ldots, y_{n-1}$ are the realizations of $Y_1, \ldots, Y_{n-1}$. His conditional estimate of V_1 given $X_1, Y_1, \ldots, Y_{n-1}$ is $E[V_1 | X_1 = x, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}]$, so his conditional expected payoff is nonnegative if and only if $x \geq y_1$. Using b^*, bidder 1 will win if and only if $X_1 > Y_1$ (recall that the event $\{X_1 = Y_1\}$ is null). Hence b^* is a best reply for bidder 1. *Q.E.D.*

THEOREM 11: *The expected price in the English auction is not less than that in the second-price auction.*

PROOF: This is identical to the proof of Theorem 8, except that $Y_2, \ldots, Y_{n-1}$ play the role of X_0. *Q.E.D.*

In effect, the English auction proceeds in two phases. In phase 1, the $n - 2$ bidders with the lowest estimates reveal their signals publicly through their bidding behavior. Then, the last two bidders engage in a second-price auction. We know from Theorem 8 that the public information phase raises the expected selling price.

By mimicking the proofs of Theorem 8 and 9, we obtain corresponding results for English auctions. Define $\bar{v}$ and $\bar{w}$ as follows.

$$\bar{v}(x, y_1, \ldots, y_{n-1}) = E\big[V_1 | X_1 = x, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}\big].$$

$$\bar{w}(x, y_1, \ldots, y_{n-1}; z) = E\big[V_1 | X_1 = x, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}, X_0 = z\big].$$

THEOREM 12: *If no information is provided by the seller, the expected price is*

$$R_N^E = E\big[\,\bar{v}(Y_1, Y_1, Y_2, \ldots, Y_{n-1}) \,|\, \{X_1 > Y_1\}\,\big].$$

If the seller announces X_0, the expected price is

$$R_I^E = E\big[\,\bar{w}(Y_1, Y_1, Y_2, \ldots, Y_{n-1}; X_0) \,|\, \{X_1 > Y_1\}\,\big].$$

Revealing information publicly raises revenues, that is, $R_I^E \geqq R_N^E$.

THEOREM 13: *In the English auction, no reporting policy leads to a higher expected price than the policy of always reporting X_0.*

6. FIRST-PRICE AUCTIONS

We begin our analysis of first-price auctions by deriving the necessary conditions for an n-tuple $(b^*, \ldots, b^*)$ to be an equilibrium point, when b^* is increasing and differentiable.[21] Suppose bidders $2, \ldots, n$ adopt the strategy b^*. If bidder 1 then observes $X_1 = x$ and bids b, his expected payoff $\Pi(b; x)$ will be given by

$$\begin{aligned}
\Pi(b; x) &= E\big[(V_1 - b)1_{\{b^*(Y_1) < b\}} \,|\, X_1 = x\big] \\
&= E\big[E\big[(V_1 - b)1_{\{b^*(Y_1) < b\}} \,|\, X_1, Y_1\big] \,|\, X_1 = x\big] \\
&= E\big[(v(X_1, Y_1) - b)1_{\{b^*(Y_1) < b\}} \,|\, X_1 = x\big] \\
&= \int_{\underline{x}}^{b^{*-1}(b)} (v(x, \alpha) - b) f_{Y_1}(\alpha \,|\, x)\, d\alpha,
\end{aligned}$$

where $\underline{x}$ is infimum of the support of Y_1. The first-order condition for a maximum of $\Pi(b; x)$ is

$$\begin{aligned}
0 &= \Pi_b(b; x) \\
&= \frac{\big(v\big(x, b^{*-1}(b)\big) - b\big) f_{Y_1}\big(b^{*-1}(b) \,|\, x\big)}{b^{*\prime}\big(b^{*-1}(b)\big) - F_{Y_1}\big(b^{*-1}(b) \,|\, x\big)},
\end{aligned}$$

where Π_b denotes $\partial\Pi/\partial b$ and F_{Y_1} is the cumulative distribution corresponding to the density f_{Y_1}. If b^* is a best reply for 1, we must have $\Pi_b(b^*(x); x) = 0$. Substituting $b^*(x)$ for b in the first-order condition and rearranging terms leads

[21] This derivation of the necessary conditions follows Wilson [**34**]. The derivation is heuristic: in general, b^* need not be continuous. For example, let $n = 2$ and take X_1 and X_2 to be either independent and uniformly distributed on [0, 1] (with probability 1/2), or independent and uniform on [1, 2]. (Note that X_1 and X_2 are affiliated.) Finally, let $V_i = X_i$. Then b^* jumps from 1/2 to 1 at $x = 1$.

to a first-order linear differential equation:[22]

$$b^{*\prime}(x) = (v(x,x) - b^*(x)) \frac{f_{Y_1}(x \mid x)}{F_{Y_1}(x \mid x)} . \tag{7}$$

Condition (7) is just one of the conditions necessary for equilibrium. Another necessary condition is that $(v(x,x) - b^*(x))$ be nonnegative. Otherwise, bidder 1's expected payoff would be negative and he could do better by bidding zero. It is also necessary that $v(\underline{x},\underline{x}) - b^*(\underline{x})$ be nonpositive. Otherwise, when $X_1 = \underline{x}$, a small increase in the bid from $b^*(\underline{x})$ to $b^*(\underline{x}) + \epsilon$ would raise 1's expected payoff from zero to some small positive number. These last two restrictions determine the boundary condition: $b^*(\underline{x}) = v(\underline{x},\underline{x})$.

THEOREM 14: *The n-tuple* $(b^*, \ldots, b^*)$ *is an equilibrium of the first-price auction, where*:

$$b^*(x) = \int_{\underline{x}}^{x} v(\alpha,\alpha)\, dL(\alpha \mid x), \qquad \textit{and} \tag{8}$$

$$L(\alpha \mid x) = \exp\left(- \int_{\alpha}^{x} \frac{f_{Y_1}(s \mid s)}{F_{Y_1}(s \mid s)}\, ds \right).^{23}$$

Let $t(x) = v(x,x)$. *Then* b^* *can also be written as*:

$$b^*(x) = v(x,x) - \int_{\underline{x}}^{x} L(\alpha \mid x)\, dt(\alpha).$$

LEMMA 1: $F_{Y_1}(x \mid z)/f_{Y_1}(x \mid z)$ *is decreasing in* z.

PROOF: By the affiliation inequality, for any $\alpha \leq x$ and any $z' \leq z$, we have $f_{Y_1}(\alpha \mid z)/f_{Y_1}(x \mid z) \leq f_{Y_1}(\alpha \mid z')/f_{Y_1}(x \mid z')$. Integrating with respect to α over the range $\underline{x} \leq \alpha \leq x$ yields the desired result. *Q.E.D.*

PROOF OF THEOREM 14: Notice that $L(\cdot \mid x)$, regarded as a probability distribution on $(\underline{x}, x)$, increases stochastically in x (that is, $L(\alpha \mid x)$ is decreasing in x). Since $v(\alpha,\alpha)$ is increasing, b^* must be increasing.

Temporarily assume that b^* is continuous in x. Then there is no loss of generality in assuming that b^* is differentiable, since Theorem 3 permits us to rescale the bidders' estimates monotonically.[24] Consider bidder 1's best response

[22] By convention, we take $f_{Y_1}(x \mid x)/F_{Y_1}(x \mid x)$ to be zero when x is not in the support of the distribution of Y_1.

[23] If the integral is infinite, $L(\alpha \mid x)$ is taken to be zero.

[24] In this proof only, we take special care to argue without assuming that the equilibrium bidding strategies are continuous or differentiable. Subsequent arguments in this paper involve a variety of differentiability assumptions that are made solely for expositional ease.

problem. It is clear that he need only consider bids in the range of b^*. Therefore, to show that $b^*(z)$ is an optimal bid when $X_1 = z$, it suffices to show that $\Pi_b(b^*(x); z)$ is nonnegative for $x < z$ and nonpositive for $x > z$. Now,

$$\Pi_b(b^*(x); z) = \frac{f_{Y_1}(x \mid z)}{b^{*\prime}(x)} \left[(v(z, x) - b^*(x)) - b^{*\prime}(x) \cdot \frac{F_{Y_1}(x \mid z)}{f_{Y_1}(x \mid z)} \right].$$

By (7), the bracketed expression is zero when $x = z$. Therefore, by Lemma 1 and the monotonicity of b^* and v, the bracketed expression (and therefore, $\Pi_b(b^*(x); z)$) has the same sign as $(z - x)$.

It remains to consider the cases where b^* (as defined by (8)) is discontinuous at some point x. That can happen only if for all positive ϵ, the first of the following expressions is infinite:

$$\int_x^{x+\epsilon} \frac{f_{Y_1}(s \mid s)}{F_{Y_1}(s \mid s)}\, ds \leq \int_x^{x+\epsilon} \frac{f_{Y_1}(s \mid x + \epsilon)}{F_{Y_1}(s \mid x + \epsilon)}\, ds$$

$$= \ln F_{Y_1}(x + \epsilon \mid x + \epsilon) - \ln F_{Y_1}(x \mid x + \epsilon);$$

the inequality follows from Lemma 1. The final difference can be infinite only if $F_{Y_1}(x \mid x + \epsilon) = 0$, and that in turn implies that $F_{Y_{n-1}}(x \mid x + \epsilon) = 0$. (Otherwise, there would be some point $z = (z_2, \ldots, z_n)$ in the conditional support of $(X_2, \ldots, X_n)$ given $X_1 = x + \epsilon$, with some $z_i < x$. By symmetry, all of the permutations of z are also in the support and therefore, by affiliation, the component-wise minimum of these permutations is in the support. But that would contradict the earlier conclusion that $F_{Y_1}(x \mid x + \epsilon) = 0$.) Thus, if any X_i exceeds x, all must.

It now follows that the bidding game decomposes into two subgames, in one of which it is common knowledge that all estimates exceed x and in the other of which it is common knowledge that none exceed x. Taking the refinement of all such decompositions, we obtain a collection of subgames, in each of which b^* is continuous. The first part of our proof then applies to each subgame separately. *Q.E.D.*

The remaining results in this section, as well as parts of the analyses in Sections 7 and 8, make use of the following simple lemma.

LEMMA 2: *Let g and h be differentiable functions for which* (i) $g(\underline{x}) \geq h(\underline{x})$ *and* (ii) $g(x) < h(x)$ *implies* $g'(x) \geq h'(x)$. *Then* $g(x) \geq h(x)$ *for all* $x \geq \underline{x}$.

PROOF: If $g(x) < h(x)$ for some $x > \underline{x}$ then, by the mean value theorem, there is some $\hat{x}$ in $(\underline{x}, x)$ such that $g(\hat{x}) < h(\hat{x})$ and $g'(\hat{x}) < h'(\hat{x})$. This contradicts (ii). *Q.E.D.*

Our first application of this lemma is in the proof of the next theorem.

THEOREM 15: *The expected selling price in the second-price auction is at least as large as in the first-price auction.*

PROOF: Let $R(x,z)$ denote the expected value received by bidder 1 if his own estimate is z and he bids as if it were x; that is, define

$$R(x,z) = E\big[\, V_1 \cdot 1_{\{Y_1<x\}} \,|\, X_1 = z\,\big].$$

Let $W^M(x,z)$ denote the conditional expected payment made by bidder 1 in auction mechanism M (in the case at hand, either the first-price or second-price mechanism) if (i) the other bidders follow their equilibrium strategies, (ii) bidder 1's estimate is z, (iii) he bids as if it were x, and (iv) he wins. For the first-price and second-price mechanisms, we have $W^1(x,z) = b^*(x)$ and $W^2(x,z) = E[v(Y_1,Y_1)\,|\,Y_1 < x,\ X_1 = z]$.

In mechanism M, bidder 1's problem at equilibrium when $X_1 = z$ is to choose a bid, or equivalently to choose x, to maximize $R(x,z) - W^M(x,z)F_{Y_1}(x\,|\,z)$. The first-order condition must hold at $x = z$:

$$0 = R_1(z,z) - W_1^M(z,z)F_{Y_1}(z\,|\,z) - W^M(z,z)f_{Y_1}(z\,|\,z), \tag{9}$$

where R_1 and W_1^M denote the relevant partial derivatives. The equilibrium boundary condition is: $W^M(\underline{x},\underline{x}) = v(\underline{x},\underline{x})$.

Clearly, $W_2^1(x,z) = 0$. From Theorem 5 it follows that $W_2^2(x,z) \geq 0$. Hence, by (9), if $W^2(z,z) < W^1(z,z)$ for some z, then $dW^2/dz = W_1^2 + W_2^2 \geq W_1^1 + W_2^1 = dW^1/dz$. Therefore, by Lemma 2, $W^2(z,z) \geq W^1(z,z)$ for all $z \geq \underline{x}$. The theorem follows upon noting that the expected prices in the first-price and second-price auctions are $E[W^1(X_1,X_1)\,|\,\{X_1 > Y_1\}]$ and $E[W^2(X_1,X_1)\,|\,\{X_1 > Y_1\}]$, respectively. *Q.E.D.*

A similar argument is used below to establish that in a first-price auction the seller can raise the expected price by adopting a policy of revealing his information.

THEOREM 16: *In the first-price auction, a policy of publicly revealing the seller's information cannot lower, and may raise, the expected price.*

PROOF: Let $b^*(\cdot\,;s)$ represent the equilibrium bidding strategy in the first-price auction after the seller reveals an informational variable $X_0 = s$. The analogue of equation (7) is:

$$b^{*\prime}(x;s) = (w(x,x;s) - b^*(x;s))\,\frac{f_{Y_1}(x\,|\,x,s)}{F_{Y_1}(x\,|\,x,s)}\,.$$

By a variant of Lemma 1, $f_{Y_1}(x\,|\,x,s)/F_{Y_1}(x\,|\,x,s)$ is nondecreasing in s, and by Theorem 5, $w(x,x;s)$ is also nondecreasing in s. The equilibrium boundary condition is $b^*(\underline{x};s) = w(\underline{x},\underline{x};s)$. Hence, applying Lemma 2 to the functions

$b^*(\cdot;s)$ for any two different values of s, we can conclude that $b^*(x;s)$ is nondecreasing in s.

Let $W^*(x,z) = E[b^*(x;X_0) \mid Y_1 < x,\ X_1 = z]$. By Theorem 5, $W_2^*(x,z) \geq 0$. Note that $W^*(\underline{x},\underline{x}) = E[w(\underline{x},\underline{x};X_0) \mid Y_1 = \underline{x},\ X_1 = \underline{x}] = v(\underline{x},\underline{x})$. If bidder 1, prior to learning X_0 but after observing $X_1 = z$, were to commit himself to some bidding strategy $b^*(x;\cdot)$, his optimal choice would be $x = z$ (since $b^*(z;x_0)$ is optimal when $X_0 = x_0$). Thus, W^* must satisfy (9). Hence, by Lemma 2, $W^*(z,z) \geq W^1(z,z)$ for all $z \geq \underline{x}$; the details follow just as in the proof of Theorem 15. The expected prices, with and without the release of information, are $E[W^*(X_1, X_1) \mid \{X_1 > Y_1\}]$ and $E[W^1(X_1,X_1) \mid \{X_1 > Y_1\}]$. Therefore, releasing information raises the expected price. *Q.E.D.*

If the seller reveals only some of his information, then, conditional on that information, $X_0, X_1, \ldots, X_n$ are still affiliated. Thus, we have the following analogue of Theorems 9 and 13.

THEOREM 17: *In the first-price auction, no reporting policy leads to a higher expected price than the policy of always reporting* X_0.

There is a common thread running through Theorems 8, 11, 12, 15, and 16 that lends some insight into why the three auctions we have studied can be ranked by the expected revenues they generate, and why policies of revealing information raise expected prices. This thread is most easily identified by viewing the auctions as "revelation games" in which each bidder chooses a *report* x instead of a bid $b^*(x)$.

No auction mechanism can determine prices *directly* in terms of the bidders' preferences and information; prices (and the allocation of the object being sold) can depend only on the reports that the bidders make and on the seller's information. However, to the extent that the price in an auction depends directly on variables other than the winning bidder's report, and to the extent that these other variables are (at equilibrium) affiliated with the winner's value estimate, the price is statistically linked to that estimate. The result of this linkage is that the expected price paid by the bidder, as a function of his estimate, increases more steeply in his estimate than it otherwise might. Since a winning bidder with estimate $\underline{x}$ expects to pay $v(\underline{x},\underline{x})$ in all of the auctions we have analyzed, a steeper payment function yields higher prices (and lower bidder profits).

In the first-price auction, for example, revealing the seller's information links the price to that information, even when the winning bidder's report x is held fixed. In the second-price auction, the price is linked to the estimate of the second-highest bidder, and revealing information links the price to that information as well. In the English auction, the price is linked to the estimates of *all* the non-winning bidders, and to the seller's estimate as well, should he reveal it. The first-price auction, with no linkages to the other bidders' estimates, yields the lowest expected price. The English auction, with linkages to all of their estimates,

yields the highest expected price. In all three auctions, revealing information adds a linkage and thus, in all three, it raises the expected price.

7. RESERVE PRICES AND ENTRY FEES

The developments in Sections 4–6 omit any mention of the seller setting a reserve price or charging an entry fee.[25] Such devices are commonly used in auctions and are believed to raise the seller's revenue. Moreover, a great deal of attention has recently been devoted to the problem of setting reserve prices and entry fees optimally (Harris and Raviv [**8**], Maskin and Riley [**11**], Matthews [**13**], Riley and Samuelson [**24**]).

It is straightforward to adapt the equilibrium characterization theorems (Theorems 6, 10, and 14) to accommodate reserve prices. In the first-price auction, setting a reserve price r above $v(\underline{x},\underline{x})$ simply alters the boundary condition, and the symmetric equilibrium strategy becomes

$$b^*(x) = r \cdot L(x^* \mid x) + \int_{x^*}^{x} v(\alpha,\alpha)\, dL(\alpha \mid x) \qquad \text{for} \qquad x \geq x^*,$$

$$b^*(x) < r \qquad \text{for} \qquad x < x^*,$$

where $x^* = x^*(r)$ is called the *screening level* and is given by

$$x^*(r) = \inf\{x \mid E[V_1 \mid X_1 = x,\ Y_1 < x] \geq r\}. \tag{10}$$

It is important to note that when the same reserve price r is used in a first-price, second-price auction, or English auction, the same set of bidders participates. Thus, in the second-price auction with reserve price r,[26] the equilibrium bidding strategy is

$$b^*(x) = v(x,x) \qquad \text{for} \qquad x \geq x^*,$$

$$b^*(x) < r \qquad \text{for} \qquad x < x^*.$$

A formal description of equilibrium with a reserve price in an English auction

[25] Actually, by permitting only nonnegative bids, we have been making the implicit assumption that there is a reserve price of zero. This reserve price has been "non-binding," in the sense that Assumption 2 (nonnegativity of V_i) ensured that no bidder would wish to abstain from participation in the auction.

If an auction is conducted with *no* reserve price, other symmetric equilibria may appear. For example, consider a first-price auction in the independent private values setting, when all $V_i = X_i$ are independent and uniformly distributed on $(0, 1)$. For every $k \geq 0$ there is an equilibrium point in which each bidder uses the bidding strategy $b(x) = (n/(n+1)) \cdot x - k/x^{n-1}$ and each has (ex ante) expected payoff $(1/n(n+1)) + k$. The range of the strategy function is $(0, n/(n+1))$ if $k = 0$, and is $(-\infty, n/(n+1) - k)$ if $k > 0$. This may explain why almost all observed auctions incorporate (at least implicitly) a reserve price.

[26] The outcome of this auction is determined as if the seller had bid r. Thus, if only one bidder bids more than r, the price he pays is equal to r. It is of interest to note that, when $v(x^*, x^*) = E[V_1 \mid X_1 = x^*, y_1 = x^*] > E[V_1 \mid X_1 = x^*,\ Y_1 < x^*]$, at equilibrium there will be no bids in a neighborhood of r.

would be lengthy; the equilibrium strategies incorporate the inference that if a bidder does not participate, his valuation must be less than x^*.

With a fixed reserve price, one can again show that the English auction generates higher average prices than the second-price auction, which in turn generates higher average prices than the first-price auction. The introduction of a reserve price does not alter these important conclusions.

More subtle and interesting issues arise when the seller has private information. If he fixes a reserve price and then reveals his information, he will generally affect x^* and hence change the set of bidders who are willing to compete. In our information revelation theorems, we assumed that the reserve price was zero, so that revealing information would not alter the set of competitors.

Given any reserve price $\bar{r}$, and realization z of X_0, let $x^*(\bar{r} \mid z)$ denote the resulting value of x^*. It is clear from expression (10) that x^* is decreasing in $\bar{r}$ and maps onto the range of X_1. Hence, there exists a reserve price $r = r(z \mid \bar{r})$ such that $x^*(r \mid z) = x^*(\bar{r})$; we call $r(z \mid \bar{r})$ the reserve price *corresponding to* z, given $\bar{r}$.

THEOREM 18: *Given any reserve price $\bar{r}$ for the first-price, second-price, or English auction, a policy of announcing X_0 and setting the corresponding reserve price raises expected revenues.*

PROOF: Let $Y_1^* = \max(Y_1, x^*(\bar{r}))$. Let $v^*(x, y) = E[V_1 \mid X_1 = x,\ Y_1^* = y]$ and let $w^*(x, y, z) = E[V_1 \mid X_1 = x,\ Y_1^* = y,\ X_0 = z]$. By Theorems 2–5, X_0, X_1, and Y_1^* are affiliated and v^* and w^* are nondecreasing, so the arguments used for Theorems 8 and 12 still apply. The argument used in the proof of Theorem 16 generalizes without difficulty. *Q.E.D.*

As with Theorems 8, 12, and 16, Theorem 18 has the corollary that no policy of partially reporting the seller's information leads to a higher expected price than full revelation: Again, "honesty is the best policy."

When both a reserve price r and an entry fee e are given, we more generally define the *screening level* $x^*(r, e)$ to be

$$x^*(r, e) = \inf\{x \mid E[(V_1 - r)1_{\{Y_1 < x\}} \mid X_1 = x] \geq e\}.$$

It is not always true that the set of bidders who will choose to pay the entry fee and participate in an auction consists of all those whose value estimates exceed the screening level. In a first-price auction, an entry fee might discourage participation by some bidder with a valuation x well above $x^*(r, e)$ if he perceives his chance of winning ($F_{Y_1}(x \mid x)$) as being slight.[27]

[27] One such case is the following. There are two variables, X_1 and X_2, so that $Y_1 = X_2$. Assume $V_1 = X_1$. With probability 1/2, the X_i's are drawn independently from a uniform distribution on [0, 2] and, with probability 1/2, from a uniform distribution on [1, 2]. Then $F_{Y_1}(x \mid x)$ jumps down from 1/2 to 1/4 as x passes up through 1. With a reserve price of zero and an entry fee of 0.32, $x^* = 0.8$ but some bidders with valuations exceeding 1.0 will choose not to bid.

If the set of bidders who participate at equilibrium in an auction with reserve price r and entry fee e does consist of those with valuations exceeding $x^*(r,e)$, then we say that the pair (r,e) is *regular* for that auction. The next result shows that among regular pairs with a fixed screening level, it pays to set high entry fees and low reserve prices, rather than the reverse.

THEOREM 19: *Fix an auction mechanism (first-price, second-price, or English), and suppose that the (reserve price, entry fee) pair (r,e) is regular. Let $(\bar{r},\bar{e})$ be another pair with the same screening level (i.e., $x^*(r,e) = x^*(\bar{r},\bar{e})$) and with $\bar{e} < e$. Then $(\bar{r},\bar{e})$ is regular, but the expected revenue from the $(\bar{r},\bar{e})$-auction is less than or equal to that from the (r,e)-auction.*

PROOF: Let $P(x,z)$ and $\bar{P}(x,z)$ denote the expected payments made by bidder 1 in the (r,e)-auction and the $(\bar{r},\bar{e})$-auction, respectively, when (i) the other bidders follow their equilibrium strategies, (ii) bidder 1's estimate is z, and (iii) he bids as if his estimate were x. (Notice that P and $\bar{P}$ are *not* conditioned on bidder 1 winning.) Defining R as in the proof of Theorem 15, we have the following equilibrium conditions: $P_1(z,z) = R_1(z,z) = \bar{P}_1(z,z)$ for all $z \geq x^*$, and $P(x^*,x^*) = R(x^*,x^*) = \bar{P}(x^*,x^*)$.

If the two auctions are first-price auctions with equilibrium strategies b and $\bar{b}$, then $P(x,z) = b(x)F_{Y_1}(x\,|\,z) + e$ and $\bar{P}(x,z) = \bar{b}(x)F_{Y_1}(x\,|\,z) + \bar{e}$. Since b and $\bar{b}$ are solutions of the same differential equation, with $b(x^*) = r < \bar{r} = \bar{b}(x^*)$, the functions cannot cross and so $b < \bar{b}$ everywhere. Also,

$$P_2(x,x) - \bar{P}_2(x,x) = \left[b(x) - \bar{b}(x)\right] \frac{\partial}{\partial z}\bigg|_{z=x} F_{Y_1}(x\,|\,z) \geq 0,$$

since the partial derivative term is negative (by affiliation). Hence, an application of Lemma 2 yields $P(z,z) \geq \bar{P}(z,z)$ for all $z \geq x^*$.

For the second-price or English auction, the payments made by a bidder when his type is z and he bids as if it were x differ only when he pays the reserve price, i.e., only when $Y_1 < x^*$. Therefore, $P_2(x,z) - \bar{P}_2(x,z) = (r - \bar{r})(\partial/\partial z)F_{Y_1}(x^*\,|\,z) \geq 0$. Once again, Lemma 2 implies that $P(z,z) \geq \bar{P}(z,z)$.

The expected payoff at equilibrium in the $(\bar{r},\bar{e})$-auction for a bidder with estimate $z \geq x^*$ is $R(z,z) - \bar{P}(z,z) \geq R(z,z) - P(z,z) \geq 0$, since (r,e) is regular. Hence, such bidders will participate in the $(\bar{r},\bar{e})$-auction and the seller's expected revenue from each of them is less than it is in the (r,e)-auction.

It remains to show that bidders with estimates $z < x^*$ will choose not to participate in the $(\bar{r},\bar{e})$-auction. In the proofs of Theorems 6, 10, and 14, we argued (implicitly) that the decision problem $\max_x R(x,z) - \bar{P}(x,z)$ is quasiconcave for each of the three auction forms, and that the maximum is attained at $x = z$. Those arguments remain valid in the present context; we shall not repeat them here. Instead, we observe this consequence of quasiconcavity: for $z < x^*$, the optimal choice of x subject to the constraint $x \geq x^*$ is $x = x^*$. The resulting expected payoff to a bidder with estimate z is $R(x^*,z) - \bar{P}(x^*,z)$.

Now, $\bar{P}(x^*,z) - P(x^*,z) = \bar{P}(x^*,x^*) - P(x^*,x^*) + (\bar{r} - r)[F_{Y_1}(x^* \mid z) - F_{Y_1}(x^* \mid x^*)]$. But $\bar{P}(x^*,x^*) = R(x^*,x^*) = P(x^*,x^*)$, and, by affiliation, the bracketed term is nonnegative. Therefore $\bar{P}(x^*,z) \geq P(x^*,z)$. Hence, the expected profit of the bidder with estimate z is $R(x^*,z) - \bar{P}(x^*,z) \leq R(x^*,z) - P(x^*,z)$, and this last expression is nonpositive because the (r,e)-auction is regular. *Q.E.D.*

8. RISK AVERSION

In the model with risk-neutral bidders, we have shown that the English, second-price, and first-price auctions can be ranked by the expected prices they generate. We have also shown that in the English and second-price auctions, the seller benefits by establishing a policy of complete disclosure of his information. In this section, we investigate the robustness of those results when the bidders may be risk averse. For simplicity, we limit attention to the case of zero reserve prices and zero entry fees.

Consider first the independent private values model, in which $V_i = X_i$ and $X_1, \ldots, X_n$ are independent. For this model, the first- and second-price auctions generate identical expected prices. Now let bidder i's payoff be $u(X_i - b)$ when he wins at a price of b, where u is some increasing, concave, differentiable function satisfying $u(0) = 0$. Let b_u^* denote the equilibrium strategy in the first-price auction. Then the analogue of the differential equation (7) is:

$$(11) \qquad b_u^{*\prime}(x) = \frac{u(x - b_u^*(x))}{u'(x - b_u^*(x))} \frac{f_{Y_1}(x)}{F_{Y_1}(x)}$$

$$\geq (x - b_u^*(x)) \frac{f_{Y_1}(x)}{F_{Y_1}(x)},$$

where the inequality follows from the concavity of u. Let b_N^* denote the equilibrium with risk-neutral bidders. From (11) it follows that whenever $b_u^*(x) \leq b_N^*(x)$, $b_u^{*\prime}(x) > b_N^{*\prime}(x)$; the equilibrium boundary condition is: $b_N^*(\underline{x}) = b_u^*(\underline{x}) = \underline{x}$. It then follows from Lemma 2 that, for $x > \underline{x}$, $b_u^*(x) > b_N^*(x)$: risk aversion raises the expected selling price. It is straightforward to verify that, with $V_i = X_i$, the second-price auction equilibrium strategy is $b^*(x) = x$, independent of risk attitudes. Thus, with independent private values and risk aversion, the first-price auction leads to higher prices than the second-price auction. In conjunction with our earlier result (Theorem 15), this implies that, for models that include both affiliation and risk aversion, the first- and second-price auctions cannot generally be ranked by their expected prices.

To treat the second-price auction when bidders are risk averse and do not know their own valuations, it is useful to generalize the definition of the function v. Let $v(x, y)$ be the unique solution of:

$$E\big[u(V_1 - v(x,y)) \,|\, X_1 = x,\ Y_1 = y\big] = u(0).$$

The proof of Theorem 6 can be directly generalized to show that $(b^*, \ldots, b^*)$ is an equilibrium point of the second-price auction when $b^*(x) = v(x,x)$.

Similarly, it is useful to generalize the definition of w. Let $w(x, y, z)$ be the unique solution of:

$$E\left[u(V_1 - w(x,y,z)) \mid X_1 = x,\ Y_1 = y,\ X_0 = z\right] = u(0).$$

In proving that releasing public information raises the expected selling price in Section 4, we used the fact that the relation

$$E\left[w(X_1, Y_1, X_0) \mid X_1, Y_1\right] \geq v(X_1, Y_1)$$

holds with equality when the bidders are risk neutral. Applied to risk-averse bidders, this inequality asserts that resolving uncertainty by releasing information reduces the risk premium demanded by the bidders. If the information being conveyed is perfect information (so that it resolves uncertainty completely), then, clearly, the risk premium is reduced to zero. But for risk-averse bidders, *it is not generally true that partially resolving uncertainty reduces the risk premium.* In fact, the class of utility functions for which any partial resolution of uncertainty tends to reduce the risk premium is a very narrow one.

Let us now rephrase this issue more formally. For a given utility function u and a random pair (V,X), define $R(x)$ by $E[u(V - R(x)) \mid X = x] = u(0)$ and define $\bar{R}$ by $E[u(V - \bar{R})] = u(0)$. We shall say that revealing X *raises average willingness to pay* if $E[R(X)] \geqq \bar{R}$.

THEOREM 20: *Let u be an increasing utility function. Then it is true for every random pair (V,X) that revealing X raises average willingness to pay if and only if the coefficient of absolute risk aversion $-u''(\cdot)/u'(\cdot)$ is a nonnegative constant.*

PROOF: We shall consider a family of random pairs (V_α, X). Let X take values in $\{0,1\}$ and let $V_\alpha = X(Z + \alpha)$, where Z is some unspecified random variable. Suppose X and Z are independent and $P\{X = 0\} = P\{X = 1\} = 1/2$. Finally, suppose $E[u(Z)] = u(0)$, and normalize so that $u(0) = 0$.

Fix u and let $\bar{R}_\alpha$ be the willingness to pay for V_α when there is no information. Let $R_\alpha(x)$ be defined as in the text. Then $R_\alpha(0) = 0$, $R_\alpha(1) = \alpha$, and $E[R_\alpha(X)] = \alpha/2$. If revealing X always increases willingness to pay, then $\bar{R}_\alpha \leqq \alpha/2$. So,

$$\begin{aligned} 0 &= E\left[u\left(V_\alpha - \bar{R}_\alpha\right)\right] \\ &= \tfrac{1}{2} E\left[u\left(Z + \alpha - \bar{R}_\alpha\right)\right] + \tfrac{1}{2} u\left(-\bar{R}_\alpha\right) \\ &\geq \tfrac{1}{2} E\left[u\left(Z + \tfrac{\alpha}{2}\right)\right] + \tfrac{1}{2} u\left(-\tfrac{\alpha}{2}\right). \end{aligned}$$

Since this holds with equality at $\alpha = 0$ and since it must hold for all α, positive

and negative, the final expression must be maximized when $\alpha = 0$:

$$
\begin{aligned}
0 &= E\big[u'(Z)\big] - u'(0), \\
0 &\geq E\big[u''(Z)\big] + u''(0).
\end{aligned} \tag{12}
$$

Now, let $g(w) = u'(u^{-1}(w))$ and let $W = u(Z)$. By varying Z, we can obtain any desired random variable W on the range of u. The conclusion reached above can be restated as: $E[W] = 0$ implies $E[g(W)] = u'(0)$. It then follows that $g(w) = cw + u'(0)$ and hence that $u'(x) = cu(x) + u'(0)$. Hence u is linear (and we are done), or $u(x) = A + Be^{cx}$. The inequality condition in (12) rules out $B > 0$; since $u' \geqq 0$, it follows that $c \leqq 0$. This proves the first assertion of the theorem.

Next fix (V, W) and let $u(x) = -\exp(-ax)$. Then

$$
\begin{aligned}
u(0) &= E\big[u(V - \bar{R})\big] \\
&= E\Big[E\Big[\exp\big(a(\bar{R} - R(X))\big)u(V - R(X)) \,|\, X\Big]\Big] \\
&= E\Big[\exp\big(a(\bar{R} - R(X))\big)E\big[u(V - R(X)) \,|\, X\big]\Big] \\
&= E\Big[\exp\big(a(\bar{R} - R(X))\big)u(0)\Big] \\
&\geqq u(0)\exp\Big[a\big(\bar{R} - E\big[R(X)\big]\big)\Big].
\end{aligned}
$$

It follows that $\bar{R} - E[R(X)] \leqq 0$. *Q.E.D.*

A straightforward corollary of this result is that $E[w(X_1, Y_1, X_0) \,|\, X_1 = x, Y_1 = y] \geqq v(x, y)$. This inequality can be used to generalize our various results concerning English and second-price auctions.

THEOREM 21: *Suppose the bidders are risk averse and have constant absolute risk aversion. Then* (i) *in the second-price and English auctions, revealing public information raises the expected price,* (ii) *among all possible information reporting policies for the seller in second-price and English auctions, full reporting leads to the highest expected price, and* (iii) *the expected price in the English auction is at least as large as in the second-price auction.*

PROOF: As in the risk-neutral developments, everything hinges on the initial statement about information release raising the expected price in a second-price auction. We shall prove only this proposition.

Note that w is a nondecreasing function. From this fact, Theorem 5, and the corollary of Theorem 20 observed in the text, we have for all $x > y$ that

$$
\begin{aligned}
v(y, y) &\leqq E\big[w(X_1, Y_1, X_0) \,|\, X_1 = y, Y_1 = y\big] \\
&= E\big[w(Y_1, Y_1, X_0) \,|\, X_1 = y, Y_1 = y\big] \\
&\leqq E\big[w(Y_1, Y_1, X_0) \,|\, X_1 = x, Y_1 = y\big].
\end{aligned}
$$

Hence $E[v(Y_1, Y_1) | \{X_1 > Y_1\}] \leqq E[w(Y_1, Y_1, X_0) | \{X_1 > Y_1\}]$, which is the desired result. *Q.E.D.*

The proof of Theorem 21 suggests that reporting information to the bidders has two effects. First, it reduces each bidder's average profit by diluting his informational advantage. The extent of this dilution is represented by the second inequality in the proof. Second, when bidders have constant absolute risk aversion, reporting information raises the bidders' average willingness to pay. This is represented by the first inequality in the proof.

Generally, partial resolution of uncertainty can either increase or reduce a risk-averse bidder's average willingness to pay. Since only an increase is possible when bidders have constant absolute risk aversion or when the resolution of uncertainty is complete, the cases of reduced average willingness to pay can only arise when the range of possible wealth outcomes from the auction is large (so that the bidders' coefficients of absolute risk aversion may vary substantially over this range) and when the unresolved uncertainty is substantial. For auctions conducted at auction houses, this combination of conditions is unusual. Thus, Theorem 21 may account for the frequent use of English auctions and the reporting of expert appraisals by reputable auction houses.

9. WHERE NOW FOR AUCTION THEORY?

The use of auctions in the conduct of human affairs has ancient roots, and the various forms of auctions in current use account for hundreds of billions of dollars of trading every year. Yet despite the age and importance of auctions, the theory of auctions is still poorly developed.

One obstacle to achieving a satisfactory theory of bidding is the tremendous complexity of some of the environments in which auctions are conducted. For example, in bidding for the development of a weapons system, the intelligent bidder realizes that the contract price will later be subject to profitable renegotiation, when the inevitable changes are made in the specifications of the weapons system. This fact affects bidding behavior in subtle ways, and makes it very difficult to give a meaningful interpretation to bidding data.

Most analyses of competitive bidding situations are based on the assumption that each auction can be treated in isolation. This assumption is sometimes unreasonable. For example, when the U.S. Department of the Interior auctions drilling rights for oil, it may offer about 200 tracts for sale simultaneously. A bidder submitting bids on many tracts may be as concerned about winning too many tracts as about winning too few. Examples suggest that an optimal bidding strategy in this situation may involve placing high bids on a few tracts and low bids on several others of comparable value (Engelbrecht-Wiggans and Weber [**6**]). Little is understood about these simultaneous auctions, or about the effects of the resale market in drilling rights on the equilibria in the auction games.

Another basic issue is whether the noncooperative game formulation of auctions is a reasonable one. The analysis that we have offered seems reasonable when the bidders do not know each other and do not expect to meet again, but it

is less reasonable, for example, as a model of auctions for timber rights on federal land, when the bidders (owners of lumber mills) are members of a trade association and bid repeatedly against each other.

The theory of repeated games suggests that collusive behavior in a single auction can be the result of noncooperative behavior in a repeated bidding situation. That raises the question: which auction forms are most (least) subject to these collusive effects? Issues of collusion also arise in the study of bidding by syndicates of bidders. Why do large oil companies sometimes join with smaller companies in making bids? What effect do these syndicates have on average prices? What forces determine which companies join together into a bidding syndicate?

Another issue that has received relatively little attention in the bidding literature concerns auctions for shares of a divisible object. Recent studies (Harris and Raviv [**8**], Maskin and Riley [**12**], Wilson [**35**]) indicate that such auctions involve a host of new problems that require careful analysis.

Much remains to be done in the theory of auctions. A number of important issues, some of which are described above, simply do not arise in the auctions of a single object that have traditionally been studied and that we have analyzed in this paper (see, for example, the survey by Weber [**31**]). Nevertheless, the treatment presented here of the role of information in auctions is a first step along the path to understanding auctions which take place in more general environments.

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APPENDIX ON AFFILIATION

A general treatment of affiliation requires several new definitions. First, a subset A of $\mathbb{R}^k$ is called *increasing* if its indicator function 1_A is nondecreasing. Second, a subset S of $\mathbb{R}^k$ is a *sublattice* if its indicator function 1_S is affiliated, i.e., if $z \vee z'$ and $z \wedge z'$ are in S whenever z and z' are.

Let $Z = (Z_1, \ldots, Z_k)$ be a random k-vector with probability distribution P. Thus, $P(A) = \text{Prob}(Z \in A)$. We denote the intersection of the sets A and B by AB and the complement of A by $\bar{A}$.

DEFINITION: $Z_1, \ldots, Z_k$ are *associated* if for all increasing sets A and B, $P(AB) \geq P(A)P(B)$.

REMARK: It would be equivalent to require $P(\bar{A}\bar{B}) \geq P(\bar{A})P(\bar{B})$ or even $P(\bar{A}B) \leq P(\bar{A})P(B)$.

DEFINITION: $Z_1, \ldots, Z_k$ are *affiliated* if for all increasing sets A and B and every sublattice S, $P(AB \mid S) \geq P(A \mid S)P(B \mid S)$, i.e., if the variables are associated conditional on any sublattice.

With this definition of affiliation, Theorems 3–5 become relatively easy to prove. However, we shall also need to establish the equivalence of this definition and the one in Section 3 for variables with densities. We begin by establishing the important properties of associated variables.

THEOREM 22: *The following statements are equivalent.*
(i) $Z_1, \ldots, Z_k$ *are associated.*

(ii) *For every pair of nondecreasing functions g and h,*

$$E[g(Z)h(Z)] \geq E[g(Z)] \cdot E[h(Z)].$$

(iii) *For every nondecreasing function g and increasing set A,*

$$E[g(Z) \mid A] \geq E[g(Z)] \geq E\big[g(Z) \mid \overline{A}\big].$$

PROOF: The inequality in (iii) is equivalent to requiring only (iii'): $E[g(Z) \mid A] \geq E[g(Z)]$, since $E[g(Z)] = P(A)E[g(Z) \mid A] + P(\overline{A})E[g(Z) \mid \overline{A}]$.

One can show that (ii) implies (iii') by taking $h = 1_A$. Similarly, to show that (iii') implies (i), take $g = 1_B$. To see that (i) implies (ii), suppose initially that g and h are nonnegative. Then we can approximate g to within $1/n$ by

$$g_n(x) = n^{-1} \sum_{i=1}^{\infty} 1_{A_{ni}}(x),$$

where $A_{ni} = \{x \mid g(x) > i/n\}$, and h can be similarly approximated using functions h_n and increasing sets B_{nj}. If $Z_1, \ldots, Z_k$ are associated, then

$$\begin{aligned} E[g_n(Z)h_n(Z)] &= n^{-2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(A_{ni}B_{nj}) \\ &\geq n^{-2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(A_{ni})P(B_{nj}) \\ &= E[g_n(Z)]E[h_n(Z)]. \end{aligned}$$

Letting $n \to \infty$ completes the proof for nonnegative g and h. The extension to general g and h is routine. *Q.E.D.*

The next result is a direct corollary of Theorem 22.

THEOREM 23: *The following statements are equivalent.*
(i) $Z_1, \ldots, Z_k$ *are affiliated.*
(ii) *For every pair of nondecreasing functions g and h and every sublattice S,*

$$E[g(Z)h(Z) \mid S] \geq E[g(Z) \mid S] \cdot E[h(Z) \mid S].$$

(iii) *For every nondecreasing function g, increasing set A, and sublattice S,*

$$E[g(Z) \mid AS] \geq E[g(Z) \mid S] \geq E\big[g(Z) \mid \overline{A}S\big].$$

Theorems 3 and 4 follow easily using part (ii) of Theorem 23, and Theorem 5 is a direct consequence of part (iii).

Finally, we verify that the present definition of affiliation is equivalent to the one given in Section 3.

THEOREM 24: *Let* $Z = (Z_1, \ldots, Z_k)$ *have joint probability density f. Then Z is affiliated if and only if f satisfies the affiliation inequality* $f(z \vee z')f(z \wedge z') \geq f(z)f(z')$ *for* μ*-almost every* $(z, z') \in \mathbb{R}^{2k}$, *where* μ *denotes Lebesgue measure.*

PROOF: If $k = 1$, both f and Z are trivially affiliated. We proceed by induction to show that if f is affiliated a.e. $[\mu]$, then Z is affiliated. Suppose that the implication holds for $k = m - 1$, and define

$Z_{-1} = (Z_2, \ldots, Z_m)$ and $z_{-1} = (z_2, \ldots, z_m)$. In the following arguments, we omit the specification "almost everywhere $[\mu]$."

Let $k = m$, and suppose that f is affiliated. Consider any two points $z_1' > z_1$. Let f_1 denote the marginal density of Z_1, and consider the function $[f(z_1', \cdot) + f(z_1, \cdot)]/[f_1(z_1) + f_1(z_1')]$, which is the conditional density of Z_{-1} given $Z_1 \in \{z_1, z_1'\}$. It can be routinely verified that this function is affiliated.[28] Therefore, by the induction hypothesis, Z_{-1} is affiliated conditional on $Z_1 \in \{z_1, z_1'\}$. Notice that, since f is affiliated, the expression $f(z_1, z_{-1})/[f(z_1, z_{-1}) + f(z_1', z_{-1})]$ is decreasing in z_{-1}. Let g be any increasing function on $\mathbb{R}^k$. Then

$$\begin{aligned} E[g(Z) \mid Z_1 = z_1] &= \frac{f_1(z_1) + f_1(z_1')}{f_1(z_1)} \cdot E\left[g(Z) \frac{f(z_1, Z_{-1})}{f(z_1, Z_{-1}) + f(z_1', Z_{-1})} \mid Z_1 \in \{z_1, z_1'\} \right] \\ &\leq \frac{f_1(z_1) + f_1(z_1')}{f_1(z_1)} \cdot E\left[\frac{f(z_1, Z_{-1})}{f(z_1, Z_{-1}) + f(z_1', Z_{-1})} \mid Z_1 \in \{z_1, z_1'\} \right] \\ &\quad \cdot E[g(Z) \mid Z_1 \in \{z_1, z_1'\}] \\ &= E[g(Z) \mid Z_1 \in \{z_1, z_1'\}], \end{aligned}$$

and it follows that $E[g(Z) \mid Z_1 = z_1] \leq E[g(Z) \mid Z_1 = z_1']$, i.e., $E[g(Z) \mid Z_1 = x]$ is increasing in x.

Now, let $h : \mathbb{R}^k \to \mathbb{R}$ also be increasing. For any non-null sublattice S, the conditional density of Z given S is $f(z) \cdot 1_S(z)/P(S)$, which is affiliated whenever f is. Also, by the induction hypothesis, Z_{-1} is affiliated conditional on Z_1. Hence

$$\begin{aligned} E[g(Z)h(Z) \mid S] &= E[E[g(Z)h(Z) \mid Z_1, S] \mid S] \\ &\geq E[E[g(Z) \mid Z_1, S] \cdot E[h(Z) \mid Z_1, S] \mid S] \\ &\geq E[g(Z) \mid S] \cdot E[h(Z) \mid S]. \end{aligned}$$

The second inequality follows from the monotonicity of $E[g(Z) \mid Z_1 = x, S]$ and $E[h(Z) \mid Z_1 = x, S]$ in x. Thus we have proved that Z is affiliated if f is.

For the converse, the idea of the proof is to take $S = \{z, z', z \vee z', z \wedge z'\}$, $A = \{x \mid x \geq z\}$, and $B = \{x \mid x \geq z'\}$, and to apply the definition of affiliation using Bayes' Theorem. This works, but is not rigorous because S is a null event. Instead, we will approximate S, AS, and BS by small but non-null events, and will then pass to the limit.

Let Q^n be the partition of $\mathbb{R}^k$ into k-cubes of the form $[i_1/2^n, (i_1 + 1)/2^n) \times \cdots \times [i_k/2^n, (i_k + 1)/2^n)$. Let $Q^n(z)$ denote the unique element of this partition containing the point z. Since $Q^0 \times Q^0$ has only countably many elements, there exists a function $q : Q^0 \times Q^0 \to \mathbb{R}$ such that (i) for every $T \in Q^0 \times Q^0$, $q(T) > 0$, and (ii) $\sum_{T \in Q^0 \times Q^0} q(T) = 1$. Define a probability measure ν on $\mathbb{R}^{2k}$ by $\nu(B) = \sum_{T \in Q^0 \times Q^0} q(T)\mu(BT)$ (recall that μ denotes Lebesgue measure). Clearly, ν is proportional to μ on every $T \in Q^n \times Q^n$, for every $n \geq 0$. Let $E^\nu[\cdot]$ be the expectation operator corresponding to ν.

Let Y and Y' be the projection functions from $\mathbb{R}^{2k}$ to $\mathbb{R}^k$ defined by $Y(z, z') = z$ and $Y'(z, z') = z'$. Y and Y' are random variables when $(\mathbb{R}^{2k}, \nu)$ is viewed as a probability space. We approximate the vector of densities $(f(z), f(z'), f(z \vee z'), f(z \wedge z'))$ by the function $X^n = (X_1^n, X_2^n, X_3^n, X_4^n)$ defined on $\mathbb{R}^{2k}$ by:

$$X^n(z, z') = E^\nu[(f(Y), f(Y'), f(Y \vee Y'), f(Y \wedge Y')) \mid (Y, Y') \in Q^n(z) \times Q^n(z')].$$

[28] The verification amounts to showing that if W_1, W_2, and W_3 are $\{0, 1\}$-valued random variables with a joint probability distribution P satisfying the affiliation inequality, then the joint distribution of W_1 and W_2 also satisfies the inequality. The conclusion follows from the inequalities:

$$(P_{111}P_{000} - P_{101}P_{010})(P_{111}P_{000} - P_{011}P_{100}) \geq 0,$$

$$P_{111}P_{001} \geq P_{101}P_{011}, \quad \text{and} \quad P_{110}P_{000} \geq P_{100}P_{010}.$$

X^n is a martingale in $\mathbb{R}^4$, and thus for almost every (z, z'),

$$\lim_{n\to\infty} X^n(z, z') = (f(z), f(z'), f(z \vee z'), f(z \wedge z'))$$

(cf. Chung [3, Theorem 9.4.8]). Also, for almost every (z, z') pair, we have $z_1 \neq z_1', \ldots, z_k \neq z_k'$. For any such pair, for sufficiently large n,

$$X^n(z, z') = 2^{nk}(P(Q^n(z)), P(Q^n(z')), P(Q^n(z \vee z')), P(Q^n(z \wedge z'))).$$

Each cube $Q^n(z)$ has a minimal element, so we may define $A_n = \{x \mid x \geq \min Q^n(z)\}$, $B_n = \{x \mid x \geq \min Q^n(z')\}$, and $S_n = Q^n(z) \cup Q^n(z') \cup Q^n(z \vee z') \cup Q^n(z \wedge z')$. The sets A_n and B_n are increasing, S_n is a sublattice, and for sufficiently large n the following three identities hold:

$$P(A_n \mid S_n) = c_n^{-1}(X_1^n + X_4^n),$$

$$P(B_n \mid S_n) = c_n^{-1}(X_2^n + X_4^n),$$

$$P(A_n B_n \mid S_n) = c_n^{-1} X_4^n,$$

where $c_n = X_1^n + X_2^n + X_3^n + X_4^n$ and each X_j^n is evaluated at (z, z'). By the definition of affiliation, we have $P(A_n B_n \mid S_n) \geq P(A_n \mid S_n) \cdot P(B_n \mid S_n)$, or equivalently, $c_n^{-1} X_4^n \geq c_n^{-2}(X_1^n + X_4^n)(X_2^n + X_4^n)$. Letting $n \to \infty$ yields (for almost every (z, z')):

$$c^{-1} f(z \vee z') \geq c^{-2}[f(z) + f(z \vee z')] \cdot [f(z') + f(z \vee z')],$$

where $c = f(z) + f(z') + f(z \vee z') + f(z \wedge z')$. A rearrangement of terms yields the affiliation inequality. *Q.E.D.*

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